

Classical & Quantum Optics

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Preface

This course presents a broad range of topics in modern classical and quantum optics. The topics are presented from an experimental point of view and are often centered around the question “How do these (quantum) effects show up in laboratory experiments?”. From that perspective, the course might have well been called “Advanced Experimental Optics”. With that name, the “Advanced Optics” would have stressed the advanced nature of the course, certainly in comparison with the first-year Bachelor course on optics. The adjective “Experimental” would have stressed the experimental point of view that the course takes towards optics. In the end, I preferred the title “Classical and Quantum Optics”

This course covers both classical and quantum optics, with a slight emphasis towards the quantum side. The classical part deals with several topics that were not covered by the first-year course on optics, which was necessarily light on mathematical tools like propagation matrices and Fourier relations. The quantum part treats the statistical properties of the optical field, associated with the quantum nature of the photon. It also covers the interaction of the optical field with simple two-level atoms and optical cavities. In order to restrict the scope of the course, which even in its present form is quite extended, we will not discuss topics like nonlinear optics, laser theory, and optics in multi-level atomic and molecular systems.

This course is based on two textbooks, both from the Oxford master series in physics, and a syllabus. The book of G. Brooker on *Modern Classical Optics* [BRO03] covers most of the topics on classical optics that I want to address. The book of M. Fox on *Quantum Optics* [FOX06] covers most of the material on quantum optics. These books are a “must have” for the course and the exercises and an asset for later reference. From each of these books I selected some hundred pages. The additional syllabus merely presents this selection and summarizes some key ideas and equations. The text is kept as short as possible; most of the course material should be derived from the two cited books, supplemented with lecture notes and articles.

This course aims to give you a certain intuition in optics. In order to reach this goal I will:

- Ask you to read the lecture material in advance.
- Keep the lectures short, focusing on the key ideas, assumptions, and physical pictures of the covered topics.
- Spend time on exercises as well, concentrating on the question “How would I solve this problem?” rather than “What is the answer?”.
- Suggest homework exercises to stimulate active participation.
- Split the final exam in two parts: after a first series of conceptual questions, which should be answered from memory and do not involve any quantitative analysis, a second series comprise the more standard quantitative exercises; these can be solved with the books at hand.

Chapter 1

Diffraction

This material is based on Chapter 3 of the book *Modern Classical Optics* [BR003]. It covers the Huygens principle (§3.2-3.6), and Fraunhofer and Fresnel diffraction (§3.7-3.17).

1.1 Linear optical systems

The propagation of light is described by Maxwell's equations. As these equations are linear in the fields, the E-field at any position inside a source-free detection volume can be written in terms of a surface integral of the incident field. We make the following simplifications. We consider:

- **Mono-chromatic light** = single frequency $\omega = kc$
- **Paraxial propagation** = angles close to the surface normal (obliquity factor $\cos \theta \approx 1$)
- **Scalar description of EM field** = single polarization

Under these assumptions we write the electric field as $E(\mathbf{r}, t) = \text{Re}[U(\mathbf{r}) \exp(-i\omega t)]$ to obtain the general linear form

$$(1.1) \quad U_{\text{det}}(\mathbf{R}) = \int K(\mathbf{R}; x, y) U_{\text{in}}(x, y) dx dy ,$$

where U_{in} and U_{det} are the incident and detected field, respectively. The Green's function or propagator $K(\mathbf{R}; x, y)$ describes the propagation from position (x, y) in the $z = 0$ plane to $\mathbf{R}$. This chapter discusses propagation through free space, after diffraction by an aperture. The next chapter discusses propagation through lens systems.

1.2 Huygens principle and diffraction

Huygens principle states that the propagation of an optical field through a plane can be described in terms of the emission of waves from Huygens secondary sources located in this plane. We apply this principle to calculate the diffraction pattern behind an aperture in an opaque screen (see Fig. 3.1 of ref. [BRO03])

$$(1.2) \quad U_{\text{dif}}(\mathbf{R}) = \int U_{\text{trans}}(x, y) \left[\frac{\exp ikr_p}{i\lambda r_p} \right] dx dy ,$$

where $k = 2\pi/\lambda$ is the wavevector and r_p is the distance from the transverse position (x, y) in the screen to the detection point $\mathbf{R}$. Equation (1.2) is a special form of the general Eq. (1.1). The term within the square brackets is the free-space propagator.

Kirchhoff boundary conditions assume that the field inside aperture is not affected by the presence of the opaque screen. Limitation to these boundary conditions become noticeable only close to the edges of the aperture at distances comparable to the optical wavelength λ . We consider two regime of diffraction: **Fraunhofer diffraction** at 'sufficiently large distance' from the aperture and **Fresnel diffraction** at smaller distances (see below).

1.3 Fraunhofer diffraction

Fraunhofer diffraction applies when the diffraction screen is illuminated with a plane wave and when the diffraction pattern is observed either in the focal plane of a lens or at 'sufficiently large distance' from the screen in the so-called far-field limit. It also applies when the screen is illuminated with a point source and lenses are used to image this source in the detection plane (see below). The Fraunhofer diffracted field is generally expressed in its angular form

$$(1.3) \quad \tilde{U}_{\text{far}}(\beta_x, \beta_y) \propto \int U_{\text{trans}}(x, y) \exp [-i (\beta_x x + \beta_y y)] dx dy ,$$

with transverse wavevector $\beta_x = k \sin \theta_x$, with $\theta_x = \lim_{r_p \rightarrow \infty} (x/r_p)$, and likewise for β_y and θ_y .

Equation (1.3) shows that the far-field diffraction pattern behind an aperture is proportional to the Fourier transform of the optical field inside this aperture. By Fourier relation, the inverse of Eq. (1.3) also applies:

$$(1.4) \quad U_{\text{trans}}(x, y) \propto \int \tilde{U}_{\text{far}}(\beta_x, \beta_y) \exp [+i (\beta_x x + \beta_y y)] d\beta_x d\beta_y ,$$

Two important examples of Fraunhofer diffraction are the diffraction behind a slit and behind a circular aperture. Using the Fourier relation of Eq. (1.3), we easily find $\tilde{U}_{\text{far}}(\beta) \propto \sin(\frac{1}{2}\beta d)/(\frac{1}{2}\beta d)$ for a slit with width d . The resulting diffraction angle from the central maximum to its first minimum yields what one might call ‘the most important result in wave optics’: $\Delta\theta = \lambda/d$. The diffraction pattern behind a circular aperture of diameter d has the more complicated form $\tilde{U}_{\text{far}}(\beta) \propto 2J_1(\frac{1}{2}\beta d)/(\frac{1}{2}\beta d)$, where J_1 is the first-order Bessel function (see Fig. 3.11 and 3.12 of ref. [BRO03]). The radius of its first ring-shaped minimum is $\Delta\theta = 1.22\lambda/d$.

1.4 Fresnel diffraction

Fresnel diffraction applies at practically any distance from the diffraction screen, whereas Fraunhofer diffraction is observable only at a ‘sufficient large’ distance. What is meant by ‘sufficient large’ can be easily determined from a Taylor expansion of the distance between two points in the source and detector plane. Restricting ourselves to displacements in the x -direction, this Taylor expansion reads

$$(1.5) \quad r_p \approx \sqrt{L^2 + (x_s - x_d)^2} \approx L - \frac{x_s x_d}{L} + \frac{x_s^2 + x_d^2}{2L},$$

where x_s and x_d are the transverse position of the source and detector and L is the on-axis distance. Insertion of this expression in Eq. (1.2) yields

$$(1.6) \quad U_{\text{dif}}(x_d) \approx \exp\left(\frac{ix_d^2}{2\lambda L}\right) \int U_{\text{trans}}(x, y) \exp\left(\frac{-i2\pi x_s x_d}{\lambda L}\right) \exp\left(\frac{i\pi x_s^2}{\lambda L}\right) \frac{dx dy}{i\lambda L}.$$

The first exponential factor in the integrand is the Fourier factor that dominates in the Fraunhofer regime. The second exponential factor is specific for the more general Fresnel regime. Its relative importance is determined by the **Fresnel number**

$$(1.7) \quad N_F \equiv \frac{(a/2)^2}{\lambda L},$$

for a circular aperture with diameter a . The Fraunhofer regime is reached when $N_F \ll 1$; the Fresnel regime corresponds to $N_F \geq 1$.

An appealing visualization of the Fresnel number in a circular symmetric system is as follows. Consider a point P_0 in the center of the diffraction pattern, look back towards the aperture, and divide the aperture plane in rings of equal distance $r_p = r_0 + m\lambda/2$, where r_0 is the on-axis distance and $m \geq 0$ is integer. **Fresnel zones** are the zones in between these rings. The Fresnel number is equal to the number of Fresnel zones that fit inside the aperture. The optical field can be concentrated on the axis by selectively blocking all the light from either the odd or the

even zones, with a so-called **Fresnel plate/lens**. The radius of consecutive rings in such a Fresnel lens scales as $\sqrt{m}$. Even a simple aperture, that passes only light from the central Fresnel zone, can increase the on-axis amplitude by a factor of 2 (= factor of 4 for intensity). To me it looked like magic in the lab when I first observed this increase in the intensity upon closing an aperture!

Fresnel diffraction often produces beautiful and intriguing interference patterns. One of the most noticeable examples is the diffraction behind a screen that covers the half space $x < 0$. Fig. 1.1 show how the intensity pattern behind the screen consists of an seemingly infinite series of bands oriented along the screen's edge. Note how the oscillations increase in spatial frequency and decrease in amplitude away from the dark-light transition at $\tilde{x} = 0$. Also note the overshoot which indicates that the presence of an opaque screen can lead to an increase of the local intensity behind that screen!

The Fresnel diffraction pattern has the same generic form at any distance behind the half screen when expressed in the dimensionless normalized transverse distance $\tilde{x} \equiv x\sqrt{2/(\lambda L)}$. This diffraction amplitude is given by the **Fresnel integral** [PED07]

$$(1.8) \quad \int_0^{\tilde{x}} e^{i(\pi/2)y^2} dy \equiv C(\tilde{x}) + iS(\tilde{x}),$$

which defines the so-called **Cornu spiral** in the complex plane (see Fig. 1.1)

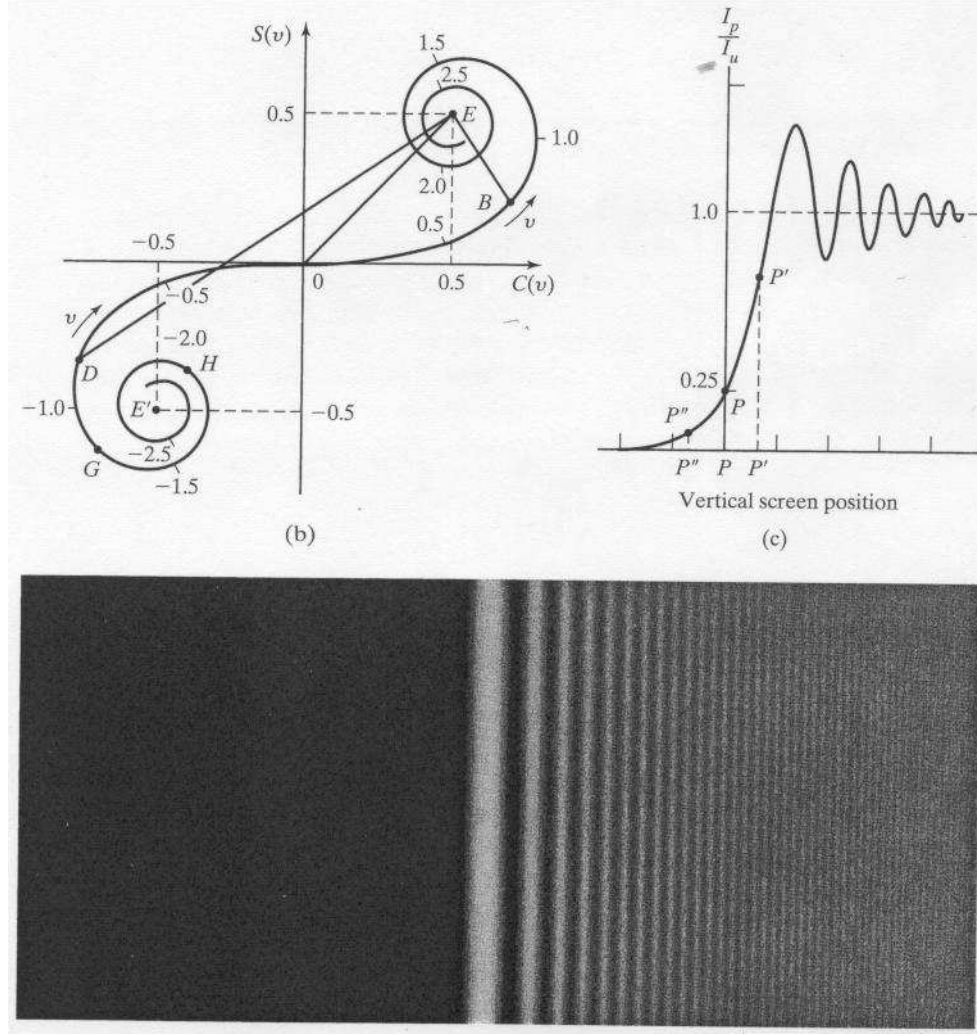


Figure 1.1: The pattern observed behind a screen that covers a half space (say $x < 0$) always has the same shape when expressed in the normalized transverse position $\tilde{x} \equiv x\sqrt{2/(\lambda L)}$. The Fraunhofer regime is unreachable as we can never get ‘sufficiently far’. Top left: Fresnel function $C(\tilde{x}) + iS(\tilde{x})$ used to calculate the intensity pattern. Top right and bottom: Intensity pattern in two different presentations (Fig. 13.14 of [PED07])

Chapter 2

Ray matrices and Gaussian beams

This material is based on Chapter 7 and part of Chapter 8 of *Modern Classical Optics* [BRO03]. It covers the matrix formulation of ray optics (§7.2-7.5), its wave optics implementation in the form of the Huygens-Kirchhoff integral (from ref. [SIE86]), a treatment of Gaussian beams (§7.6-7.9 and 8.2), and a brief description of optical cavities (§8.3-8.5).

2.1 Matrix formulation of ray optics

We consider the propagation of an optical ray in a 2D sheet and characterize the ray by its transverse coordinate x and angle θ , where $dx/dz = \tan \theta$ and $\tan \theta \approx \theta$ in the paraxial regime. The propagation through any linear optical system can be described by the matrix multiplication

$$(2.1) \quad \begin{pmatrix} x \\ \theta \end{pmatrix}_{out} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}_{in}.$$

We will limit ourselves to rays that begin and end in air/vacuum. The more general case is discussed in ref. [BRO03].

Most optical systems comprise a series of two common components, being either free-space propagation over a distance L or focussing/de-focussing with a lens or curved mirror with focal length f . The **ray transfer matrices** of these composite systems can be easily constructed from a multiplication of the matrices

$$(2.2) \quad M_L = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad M_f = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}.$$

Note that $\text{Det}(M) = 1$ for any system comprised of these two elements. This property signifies the conservation of phase space $\Delta x \cdot \Delta \theta$, which we will later denote as the **optical etendue**.

An important composite system is the so-called $2f$ system, which comprises two sections of free-space propagation over a distance f positioned around a positive lens with focal length f . The total ray matrix of this system is

$$(2.3) \quad M_{2f} = \begin{pmatrix} 0 & f \\ -1/f & 0 \end{pmatrix}.$$

By combining two of these systems we can construct a $4f$ system, which is a telescope with magnification -1 as $M_{2f}^2 = -1$.

The $2f$ system is often used to create a full Fourier transformation of an optical field. An approximate Fourier transform can be produced with the more general Lf system, where the first propagation is over a distance L instead of f and where

$$(2.4) \quad M_{Lf} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & f \\ -1/f & 1 - (L/f) \end{pmatrix}.$$

The matrix element $D = 1 - (L/f) \neq 0$ quantifies how much this system deviates from the ideal Fourier system. In the focal plane of the lens we still have the proper (Fourier-type) relation $x_{\text{out}} = f\theta_{\text{in}}$, but the ray angle θ_{out} now depends both on x_{in} and θ_{in} .

2.2 Huygens-Kirchhoff integral for wave optics

The matrix formulation of ray optics has a counterpart in wave optics. This so-called **Huygens-Kirchhoff integral** formulation of wave optics, which we'll cite without proof, reads [SIE86]

$$(2.5) \quad E_{\text{out}}(x) = \int K(x, x') E_{\text{in}}(x') dx',$$

where x' and x are transverse positions in the source and detection plane, respectively, and where the 2D integration Kernel

$$(2.6) \quad K(x, x') = \frac{1}{\sqrt{i\lambda B}} \exp\left(ik \frac{Dx^2 - 2xx' + Ax'^2}{2B}\right).$$

The funny-looking normalization by $\sqrt{i\lambda B}$ for the considered 2D-sheet changes to the more common normalization by $(i\lambda B)$ in 3D.

Equations (2.5) and (2.6) show that the coefficients of the $ABCD$ matrix in ray optics also determine the propagation of the (field profile of the) optical wave from the source plane to the detection plane! Again we note that the Lf system introduced above only produces an approximate Fourier transform in its focal plane; the intensity profile is correct, as it corresponds to the square absolute value of the Fourier transform of the input field, but the phase front has an additional curvature if $L \neq f$, such that $D \neq 0$.

Note the strong resemblance between the integration Kernel of Eq. (2.6) and the free-space propagator of Chapter 1. The resemblance is complete when we interpret B as an effective propagation length and use the quadratic Taylor expansion $r_p(x, x') - r_0 \approx (Dx^2 - 2xx' + Ax'^2)/2B$. At $A = D = 1$ and $B = L$ the Huygens-Kirchhoff integral reduces to the Fresnel integral of Eq. (1.6). The one-to-one imaging system with $A = D = -1$ and $B = 0$ results in $E_{\text{out}}(x) \propto E_{\text{in}}(-x)$. For the $2f$ system with $A = D = 0$ and $B = f$, the above equation corresponds to a Fourier relation between the field profile in the source and detection plane, as encountered in Fraunhofer diffraction.

2.3 Gaussian beams

Gaussian beams are characterized by their minimum beam width or **beam waist** w_0 and its position (often defined as $z = 0$). Figure 7.5 of ref. [BRO03] shows how the beam width changes with propagation as

$$(2.7) \quad w(z) = w_0 \sqrt{1 + (z/z_0)^2},$$

where $z_0 = \frac{1}{2}kw_0^2 = \pi w_0^2/\lambda$ is the so-called **Rayleigh range** or **confocal parameter**. For easy reference, we note that a (near-field) intensity pattern $I(x) \propto \exp(-2x^2/w_0^2)$ (FWHM = $\sqrt{2 \ln 2} w_0 \approx 1.18 w_0$) corresponds to a far-field intensity $\tilde{I}(\theta) \propto \exp(-2\theta^2/\theta_0^2)$ with $\theta_0 = \lambda/(\pi w_0)$. The complex optical field of a fundamental Gaussian beam can be written as

$$(2.8) \quad E(r, z, t) = \frac{1}{q} \exp\left(\frac{ikr^2}{2q}\right) \exp i(kz - \omega t),$$

where the Gaussian beam parameter q is given by

$$(2.9) \quad \frac{1}{q} = \frac{1}{R} + \frac{i\lambda}{\pi w^2},$$

with $R(z) = z + z_0^2/z$ the radius of curvature and $w(z)$ the beam width.

Upon propagation through an ABCD system, the Gaussian beam parameter changes as

$$(2.10) \quad q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}.$$

Free-space propagation of a beam with a waist w_0 positioned at $z = 0$ results in $q = z - iz_0$. This elegant form is sometimes denoted as the 'complex source' description of wave propagation.

Next we compare the evolution of the on-axis ($r = 0$) optical phase of a Gaussian beam with that of a plane wave. For the fundamental TEM_{00} Gaussian beam the difference between the two is easily found by rewriting the pre-factor ($1/q$) in Eq. (2.8) as

$$(2.11) \quad \frac{1}{q} = \frac{1}{z - iz_0} = \frac{i}{\sqrt{z^2 + z_0^2}} \exp\left(-i \arctan \frac{z}{z_0}\right).$$

The extra phase variation $\alpha(z) = -\arctan z/z_0$ is called the **Gouy phase**; it amounts to a phase lag of $\pi/2$ for propagation from the focal point to the far-field or π for propagation from $z = -\infty$ to $+\infty$. This phase lag basically results from the reduction of the wave vector in the forward direction $k_z = \sqrt{k^2 - k_{tr}^2}$ in the presence of transverse momentum (transverse wave vector k_{tr}). For the higher-order TEM_{nm} modes this phase lag increases to $\pi(n + m + 1)$ from $z = -\infty$ to $+\infty$.

Higher-order Gaussian beams are described by Eq. (8.1) of ref. [BRO03], which we rewrite as

$$(2.12) \quad \begin{aligned} E_{nm}(r, z, t) = & \frac{i}{\sqrt{z^2 + z_0^2}} \exp\left(\frac{-r^2}{w^2} + \frac{ikr^2}{2R}\right) \exp i(kz - \omega t) \\ & H_n\left(\frac{\sqrt{2}x}{w}\right) H_m\left(\frac{\sqrt{2}y}{w}\right) \exp\left[-i(n + m + 1)\text{atan}\left(\frac{z}{z_0}\right)\right], \end{aligned}$$

where $H_0(\xi) = 1$, $H_1(\xi) = 2\xi$, $H_2(\xi) = 4\xi^2 - 2$, etc. are the physicists' Hermite polynomials. The optical-field profiles are identical to those of the higher-order quantum state $\psi_n(x)$ of an harmonic oscillator. The associated intensity profiles, as depicted on page 172 and 173 in ref.[BRO03], correspond to the quantum probabilities $|\psi_n(x)|^2$ of the harmonic-oscillator states. The mean-square (intensity-weighted) width of the described HG_{nm} profile is $\langle x^2 \rangle = w^2(n + \frac{1}{2})$ and $\langle y^2 \rangle = w^2(m + \frac{1}{2})$.

2.4 Optical cavities

Gaussian beams are the natural eigenmodes of optical cavities with curved spherical mirrors. For a two-mirror (= **Fabry-Perot**) cavity, the Gaussian waist w_0 and

its position $z = 0$ can be found by setting the beam curvature $R(z)$ equal to the mirror curvature at the two mirrors. Solutions can only be found for stable cavity configurations, where

$$(2.13) \quad 0 < (1 - L/R_1)(1 - L/R_2) < 1,$$

where L is the cavity length and R_1 and R_2 are the mirror curvatures (see Fig. 8.1 of ref. [BRO03]). These three parameters determine the beam-waist w_0 , the frequency spacing between consecutive longitudinal modes $\Delta\nu = c/(2L)$, and the transverse mode spacing $\Delta\nu_{\text{trans}}/\Delta\nu_{\text{long}} = [\text{atan}(z_1/z_0) - \text{atan}(z_2/z_0)]/\pi$, where $z_0 = \pi w_0^2/\lambda$ and z_1 and $z_2 = z_1 + L$ are the positions of the mirrors with respect to the beam waist. For symmetric cavities ($R_1 = R_2 = R$), we obtain the obvious $z_2 = -z_1 = L/2$ and

$$(2.14) \quad z_0 = \frac{1}{2}\sqrt{L(2R - L)},$$

with $z = R/2$ for the confocal ($L = R$) Fabry-Perot cavity.

Chapter 3

Optics in multi-layered systems

The first part of this chapter is based on chapter 6 of *Modern Classical Optics* [BRO03]. It covers a description of optical reflection and transmission through a layered system in terms of the optical impedance and the transfer matrix as well as its application to anti-reflection coatings and dielectric mirrors (§6.1-6.8). The second part introduces a different type of transfer matrix, described a.o. in the book *Optical waves in layered media* of Yeh [YEH05]. It also introduces a technique to calculate the eigenmodes of a planar optical waveguide.

3.1 Optical impedance & reflection

Maxwell's equations show that the ratio between the electric and magnetic field components of an optical plane wave in a uniform medium depends only on material properties. This ratio is the so-called **optical impedance**, defined as

$$(3.1) \quad Z \equiv \left| \frac{E}{H} \right| = \frac{1}{n} \sqrt{\frac{\mu_0}{\epsilon_0}} \equiv \frac{Z_0}{n},$$

where $Z_0 = 120\pi \Omega \approx 377 \Omega$ is the optical impedance of vacuum and where the material was assumed to be non-magnetic ($\mu_r = 1$). The dimension Ω arises naturally from the ratio of the dimensions V/m for the E -field and A/m for the H -field.

Optical reflection is a natural consequence of impedance mismatch. This statement, which applies to any wave phenomenon, can be easily quantified for the reflection and transmission of an optical plane wave incident on the interface between medium 1 and 2. We decompose the wave in medium 1 in the incident and reflected wave, with relative amplitudes 1 and r_{12} and denote the amplitude of the transmitted wave in medium 2 by t_{12} . The required continuity of (the parallel components

of) E and H now results in two equations: $1 + r_{12} = t_{12}$ and $(1 - r_{12})/Z_1 = t_{12}/Z_2$, where the minus sign is linked to the inversion of the propagation direction. An easy rewrite yields

$$(3.2) \quad r_{12} \equiv \frac{E_{1L}}{E_{1R}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad t_{12} \equiv \frac{E_{2R}}{E_{1R}} = \frac{2Z_2}{Z_1 + Z_2} \quad \text{or} \quad \left(\frac{\cos \theta_1}{\cos \theta_2} \right) \frac{2Z_2}{Z_1 + Z_2},$$

where (E_{1L}, E_{1R}) and (E_{2L}, E_{2R}) denote the amplitudes of the leftward and rightward propagating traveling waves in medium 1 and 2, respectively. These equations apply to normal incidence ($\theta = 0$), but can also be used at non-normal incidence if we interpret $Z \equiv E_{\parallel}/H_{\parallel}$ and distinguish between two optical polarizations. For TM (= transverse magnetic) polarization, also denoted as p-polarization, the optical impedance $Z_{TM} = Z_p \equiv |E_{\parallel}/H_{\parallel}| = Z_0 \cos \theta / n$. For TE (= transverse electric) polarization, also denoted as s-polarization, the optical impedance $Z_{TE} = Z_s \equiv |E_{\parallel}/H_{\parallel}| = Z_0 / (n \cos \theta)$. The subscript s is derived from the German word 'Senkrecht' = perpendicular (E -fields perpendicular to this plane). The factor $\cos \theta_1 / \cos \theta_2$ in the expression for t_{12} is present only for TM-polarized light and translates $E_{\parallel}$ into E . Note that Eqs. (3.2) are in agreement with the Stokes relations $r_{21} = -r_{12}$ and $t_{12}t_{21} = 1 - |r_{12}|^2$.

3.2 Transfer matrix in a layered system

The reflection and transmission coefficients of a multi-layered structure can be calculated by keeping track of the amplitudes of the forward and backward propagating EM fields in all media. The amplitudes are linked by the required continuity of $E_{\parallel}$ and $H_{\parallel}$ at each interface and the fixed E/H -ratio in each medium. The mentioned continuity relations allow one to calculate the reflection and transmission amplitudes of any multi-layered structure by bookkeeping via multiplication of 2x2 matrices. Brooker [BRO03] expresses the EM waves in terms of the total electric and magnetic field components parallel to the interface. His 2-vector $(E_{\parallel}(z), Z_0 H_{\parallel}(z))$ is thus continuous across the interface, but varies upon propagation as

$$(3.3) \quad \begin{pmatrix} E_{\parallel}(-l) \\ Z_0 H_{\parallel}(-l) \end{pmatrix} = \begin{pmatrix} \cos kl & -(i/n) \sin kl \\ -in \sin kl & \cos kl \end{pmatrix} \begin{pmatrix} E_{\parallel}(0) \\ Z_0 H_{\parallel}(0) \end{pmatrix},$$

for a layer thickness l and refractive index n . Multiplication of a series of such matrices yields the **transfer matrix** of the complete system. This multiplication is generally written down from right to left, i.e., starting from the outgoing wave where $E_{\parallel}/H_{\parallel} = Z_0/n_{\text{out}}$. Equation (3.3) applies to illumination at normal incidence, but can be extended to arbitrary angles by replacing n by the more general $Z_0/Z_{\parallel}$. The internal angles are related by Snell's law ($n_i \sin \theta_i$ is constant).

The beauty of the matrix multiplication lies in the fact that it allows one to replace all media to the right of a given plane by a single medium with an effective optical impedance, as the only property that really matters is the ratio $Z \equiv |E_{\parallel}/H_{\parallel}|$ at the mentioned plane. For stacks of layers with optical thicknesses that are only multiples of $\lambda/4$, the impedance of the full stack can be easily found by remembering the following rule: the additional of a quarter-wave layer with impedance Z_{layer} on top of a stack with effective impedance Z_{load} changes the stack impedance to

$$(3.4) \quad Z = Z_{\text{layer}}^2 / Z_{\text{load}} .$$

This simple equation has many implications. It for instance implies that a single $\lambda/4$ layer acts as a perfect anti-reflection coating if $Z_{\text{layer}}^2 = Z_{\text{load}} Z_{\text{in}}$, which translates into $n_{\text{layer}} = \sqrt{n_{\text{out}}}$ for a plate with index n_{out} in air. It also implies that additional $\lambda/2$ layers of any material doesn't modify the overall reflection and transmission. Finally, it implies that the optical impedance of a stack of two $\lambda/4$ layers with optical impedances Z_2 and Z_3 on top of a structure with impedance Z_4 is $Z_{\text{tot}} = (Z_2/Z_3)^2 Z_4$.

3.3 Alternative definition of transfer matrices

Most textbooks introduce a different kind of transfer matrix, which is based on a separation of the total optical field in its forward- and backward-travelling waves, instead of its E and H field components. The advantage of this alternative description is that it presents a more natural physical picture: propagation is described by simple phase factors and reflection and transmission occurs at the interfaces. In this alternative description, the electric field in each medium i is separated into a forward-propagating component E_{iR} and a backward-propagating component E_{iL} (R =right and L =left). The propagation through a layer with thickness d_i and index n_i is now described by a propagation matrix P_i such that

$$(3.5) \quad \begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix}_{\text{left}} = P_i \begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix}_{\text{right}} = \begin{pmatrix} \exp(-ik_i d_i) & 0 \\ 0 & \exp(ik_i d_i) \end{pmatrix} \begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix}_{\text{right}} ,$$

where $k_i \equiv \sqrt{n_i^2 k_0^2 - k_{\parallel}^2}$ is the wave vector component perpendicular to the interface. The reflection from a single interface between medium i and j is described by the reflection matrix M_{ij} , where

$$(3.6) \quad \begin{pmatrix} E_{iR} \\ E_{iL} \end{pmatrix} = M_{ij} \begin{pmatrix} E_{jR} \\ E_{jL} \end{pmatrix} = \begin{pmatrix} 1 + (Z_i/Z_j) & 1 - (Z_i/Z_j) \\ 1 - (Z_i/Z_j) & 1 + (Z_i/Z_j) \end{pmatrix} \begin{pmatrix} E_{jR} \\ E_{jL} \end{pmatrix} .$$

The reflection matrix of a single interface is symmetric and can also be written as

$$(3.7) \quad M_{12} = \frac{1}{t_{12}} \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix} ,$$

where r_{12} and t_{12} are the amplitude reflection and transmission coefficients for light propagating from medium 1 to 2. The reflection and transmission coefficients r_{21} and t_{21} of the counter-propagating wave are found by substitution. The resulting Stokes relations $r_{21} = -r_{12}$ and $t_{12}t_{21} = 1 - |r_{12}|^2$ are also valid in the presence of absorption and at a non-zero angle of incidence.

The optical transfer through an arbitrary stack of layers can be calculated by multiplication of a series of M and P matrices. As an example we consider the transmission from medium 1 via medium 2 to medium 3 as[YEH05]

$$(3.8) \quad \begin{pmatrix} E_{1R} \\ E_{1L} \end{pmatrix} = M_{13} \begin{pmatrix} E_{3R} \\ E_{3L} \end{pmatrix} = M_{12} P_2 M_{23} \begin{pmatrix} E_{3R} \\ E_{3L} \end{pmatrix}.$$

After matrix multiplication we find among others the familiar transmission characteristics of a single slab of thickness d (=Fabry-Perot resonator):

$$(3.9) \quad t_{13} = t_{12}t_{23}e^{i\phi} / (1 + r_{12}r_{23}e^{i2\phi}),$$

where $\phi \equiv k_2d$ is the phase delay acquired during a round trip in the slab.

For any stack of non-absorbing layers a more general set of Stokes relation can still be derived from the **reversibility** of optical waves. This more general Stokes relations read $t_{ij}t_{ji}^* + r_{ij}r_{ji}^* = 1$ and $t_{ij}r_{ji}^* + t_{ji}^*r_{ij} = 0$. [YEH05] These relations break down if any of the layers is absorptive. For a stack that begins and ends in media with the same index of refraction n , the transmission amplitudes for propagation from left-to-right and right-to-left are equal, i.e. $t_{ij} = t_{ji}$. This symmetry doesn't necessary apply to the reflection amplitudes of stacks that contain absorbing layers, where e.g. $r_{31} \neq r_{13}$ for a stack that starts with a highly reflective mirror and ends with a strong absorber.

The relation $t_{ij} = t_{ji}$ is a special case of the more general **principle of reciprocity**, which states that “the ratio of the optical amplitude at the detector divided by that at the source doesn't change if we swap the positions of the source and detector”. Reciprocity differs from time reversal symmetry, as time reversal changes diverging into a converging waves whereas reciprocity changes the role of the emitter and receiver. Reciprocity is a natural consequence of the exchange symmetry of the field propagator $K(\mathbf{r}_1, \mathbf{r}_2) = K(\mathbf{r}_2, \mathbf{r}_1)$. It can only be broken by the presence of a DC magnetic field in a material with a Faraday effect.

3.4 Distributed Bragg Reflector (DBR)

Figure 3.1 shows a popular layer structure, comprising an alternating stack of $\lambda/4$ layers of medium 2 and 3 on top of a substrate made of medium 4. This structure

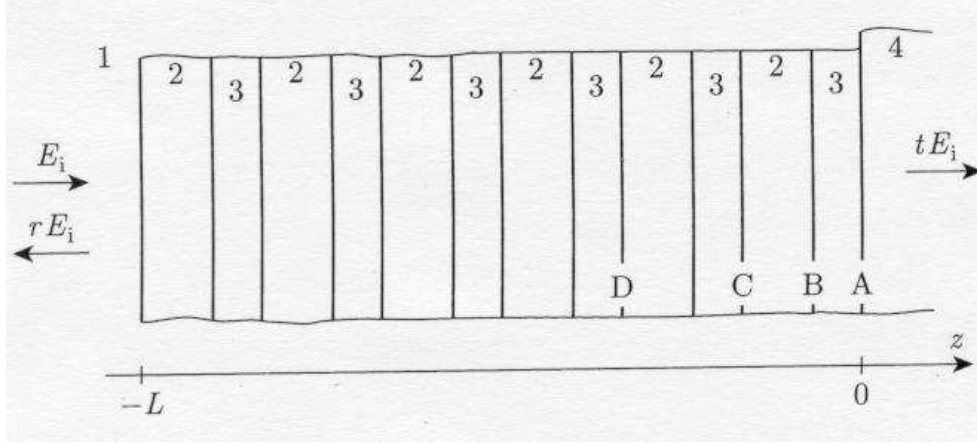


Figure 3.1: An example of a layered structure. The amplitude reflection coefficient r and transmission coefficient t of the complete structure can be calculated via a multiplication of 2×2 matrix that describe the transmission and reflection properties of the individual layers and interfaces. High-reflectivity mirrors can be made from a stack of $\lambda/4$ layers of two alternating media with high and low refractive index ($n_2 d_2 = n_3 d_3 = \lambda_0/4$). (Fig. 6.4 of [BRO03])

is so popular because it acts as a high-reflectivity mirror if the stack is thick enough and the index contrast n_2/n_3 is large enough. The explanation is simple: as every pair of layers modifies the optical impedance of the stack by a factor $(Z_2/Z_3)^2$, the total impedance of a stack of p layers $Z_{\text{tot}} = (Z_2/Z_3)^{2p} Z_4$ will either decrease to zero or increase to infinity for $p \rightarrow \infty$. In both case, the corresponding intensity reflection $R \equiv |r|^2$ will approach 1 via

$$(3.10) \quad (1 - R) \propto \left(\frac{n_{\text{lo}}}{n_{\text{hi}}} \right)^{2p},$$

where $n_{\text{lo}} = n_2$ and $n_{\text{hi}} = n_3$ if $n_2 < n_3$ and vice versa. A high reflectivity is reached with fewer layer if the index contrast $n_{\text{hi}}/n_{\text{lo}}$ is large. It also helps to optimize the index contrast at the 12 and 34 interface, by taking $n_2 > n_3$ if $n_4 > n_1$. High-reflectivity mirrors that are based on this concept are called **Distributed Bragg Reflectors (DBRs)** to indicate that the high reflectivity originates from the constructive interference between the reflections at all individual interfaces via what is generally denoted as the Bragg condition.

What happens to the reflectivity of a DBR mirror if we tune the optical frequency away from its resonance condition $k_2 l_2 = k_3 l_3 = \pi/2$? The answer can be found by straight-forward matrix multiplication, but the result is messy. The key result is best illustrated in Fig. 3.2, which shows the generic behavior of the reflectivity as a function of frequency detuning for four different stacks with a relatively

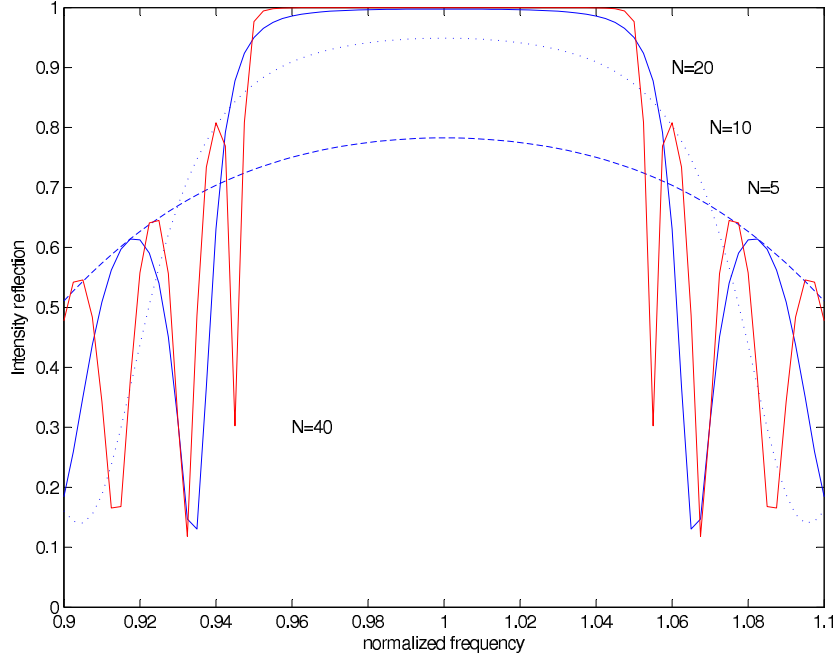


Figure 3.2: Intensity reflection versus normalized frequency ω/ω_0 of four different DBR structures, comprising $N = 5, 10, 20$ and 40 pairs of layers ($n_2 = 3.5$, $n_3 = 3.0$) on top of a $n_4 = 3.5$ substrate ($n_1 = 1$). Note how the stopband builds up for increasing N .

small index contrast ($n_2/n_3 = 3.5/3.0$ in this figure). For thick stacks, the intensity reflectivity R remains close to one for small detunings but decreases rapidly beyond a critical detuning (half width) of

$$(3.11) \quad \frac{\Delta\omega}{\omega} \approx \pm \left(\frac{n_3 - n_2}{n_3 + n_2} \right) \left(\frac{2}{\pi} \right)$$

The central region is called the **stopband** to indicate that the transmission $T = 1 - R$ is practically zero at these optical frequencies. Inside the stopband, the phase of the reflected light changes as $\phi \approx 2\Delta\bar{k}d_{\text{pen}}$, where $\Delta\bar{k} = \frac{1}{2}(n_2 + n_3)\Delta\omega/c$ and

$$(3.12) \quad d_{\text{pen}} = \frac{\lambda_0}{4|n_3 - n_2|},$$

is the effective penetration depth of the optical intensity into the DBR.[BRO95] The angle and frequency dependent reflection of a DBR is similar to that of a fixed mirror positioned at a distance d_{pen} behind the front facet.

3.5 Planar optical waveguides

The matrix formalism described in this chapter, and in particular the approach based on the forward and backward travelling waves, can also be used to find the dispersion relation and transverse mode profiles of the guided modes of multi-layered slab wave guides. These eigenmodes correspond to combinations of the optical frequency ω and parallel wavevector $k_{\parallel}$ for which the ratio r_{13}/t_{13} diverges [JOA08, YEH05], i.e., for which there can be a reflected wave without any input! The same eigenmodes can be found by “cutting-and-gluing” of forward and backward traveling waves in each of the media, under the restriction that the two outer media contain only outward-propagating waves.[YEH05]

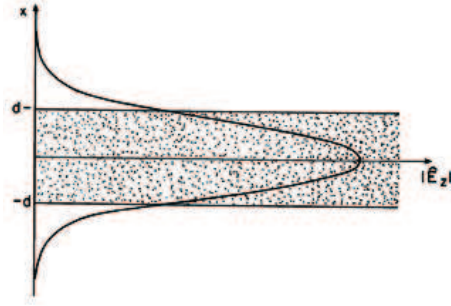


Figure 3.3: *The optical field of the guided mode in a thin slab is cosine-shaped inside the slab and decays exponentially outside the slab.*

Consider for instance a single layer of thickness d_2 and index n_2 sandwiched between two semi-infinite media 1 and 3. The reflection coefficients r_{13} and r_{31} of this structure diverge when ω and $k_{\parallel}$ are chosen such that the optical field forms a standing wave in medium 2 that turns smoothly into exponential decays in the outer media. This condition requires $n_2 > k_{\parallel}c/\omega > \{n_1, n_3\}$. The dispersion relation $(k_{\parallel}, \omega)$ is different for TE and TM polarized waves, the former being more confined (with larger $k_{\parallel}$) than the later [JOA08, YEH05]. The dispersion relation of the TE-polarized eigenmode in a symmetric ($n_1 = n_3$) slab waveguide of thickness d is [YEH05]

$$(3.13) \quad \tan(k_{\perp}d) = \frac{2k_{\perp}q}{k_{\perp}^2 - q^2}$$

where $k_{\perp} = \sqrt{(n_2\omega/c)^2 - k_{\parallel}^2}$ is the perpendicular wavevector in the slab and $q = \sqrt{k_{\parallel}^2 - (n_1\omega/c)^2}$ is the inverse penetration length in the surrounding media. The dispersion relation of the TM-polarized eigenmode is found by replacing q by $(n_2/n_1)q$.

Strangely enough, even the reflectivity of a single interface can diverge for certain combinations of $(\omega, k_{\parallel})$, but only for the interface between a dielectric and a lossless metal and only for TM waves. The resonance condition $Z_1 + Z_2 = 0$ corresponds to the excitation of surface plasmon polaritons.

Chapter 4

Coherence in optics

This material is based on chapters 9 and 10 of the book *Modern Classical Optics* [BRO03]. It discusses the importance of temporal and spatial coherence in optics.

4.1 Introduction to optical coherence

The theory of optical coherence describes the properties of optical fields of which the amplitude or phase vary in time, i.e., of fields $E(\mathbf{r}, t) = \text{Re}[U(\mathbf{r}, t) \exp i(\mathbf{k}_0 \mathbf{r} - \omega_0 t)]$ with time-dependent complex amplitude $U(\mathbf{r}, t)$. This description requires a quantitative treatment of the statistical properties of a randomly-varying optical field. It requires 'great care in order to avoid common misconceptions' [BRO03] and can become highly mathematical. Brooker keeps the mathematics light and refers a.o. to the book of "Optical Coherence & Quantum Optics" of Mandel and Wolf [MAN95] for a more complete description.

A key concept in the description is the notion that incoherence is a consequence of randomness and is basically a matter of time scales. Light can be quite coherent on short timescales while being incoherent on timescales long enough to average over the natural fluctuations. Most quantitative descriptions of coherence revolve around the **cross-correlation function** of the complex optical field

$$(4.1) \quad \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \equiv \langle U^*(\mathbf{r}_1, t) U(\mathbf{r}_2, t + \tau) \rangle_t \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} U^*(\mathbf{r}_1, t) U(\mathbf{r}_2, t + \tau) dt,$$

where the brackets $\langle \rangle_t$ denote time averaging. Coherence is generally defined in terms of what one would observe with a 'sufficiently slow' detector ($T \rightarrow \infty$).

The description of randomly varying fields is subtle, because these fields don't have a natural start or end but keep on fluctuating. These fields are outside the

class of ‘quadratically integrable functions’ that are generally introduced for Fourier transformations. As the time-integrated power (=energy) in such randomly-varying field diverges, we instead can only talk about the average power or expected energy *per unit time*. These fields require a special normalization for their Fourier transformation. One approach could be to include the time duration in the definition of the Fourier transform as

$$(4.2) \quad \tilde{U}(\omega) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} U(t) \exp(i\omega t) dt.$$

The $1/\sqrt{T}$ pre-factor results in somewhat funny units for $\tilde{U}(\omega)$, such as “... per $\sqrt{Hz}$ ”. Although properly normalized, the Fourier-transformed field $\tilde{U}(\omega)$ is still as random as the time-domain field $U(t)$. It is thus easier to transform the auto-correlation function instead, via

$$(4.3) \quad |\tilde{U}(\omega)|^2 = \int_{-\infty}^{\infty} \langle U(t)U(t+\tau) \rangle_t \exp(i\omega\tau) d\tau.$$

The spectrum $|\tilde{U}(\omega)|^2$ represents the “average square field per unit frequency bandwidth”, i.e., the contribution of a specific frequency range to the average mean-square field $\langle |U(t)|^2 \rangle_t$. Equation (4.3) is the so-called **Wiener-Khintchine theorem**. It applies to **stationary random fields**, i.e., to fields for which the expectation values do not change with time (see also §10.6 of ref. [BRO03]).

4.2 Quantitative treatment of optical coherence

Optical coherence is an essential requirement for any interference experiment. If the incident field is not sufficiently coherent the resulting interference pattern will vary in time and the interference fringes will wash out and become invisible after time-averaging. As a prototype interference experiment, we consider Young’s double slit and write the intensity at sufficiently large distance behind two small slits as

$$(4.4) \quad I \propto |E_1 + E_2 \exp(i\varphi)|^2 = |E_1|^2 + |E_2|^2 + 2\text{Re}[E_1^* E_2 \exp(i\varphi)],$$

where φ is the phase difference associated with the difference in optical path length from the detection point to slit 1 or 2 and $E = U$ is the complex optical field. For identical slits ($|E_1|^2 = |E_2|^2$) the visibility of the interference fringes is determined by the normalized cross-correlation function

$$(4.5) \quad \gamma_{12}(\tau) \equiv \gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \equiv \frac{\langle E^*(\mathbf{r}_1, t) E(\mathbf{r}_2, t + \tau) \rangle_t}{\sqrt{\langle |E(\mathbf{r}_1, t)|^2 \rangle_t \langle |E(\mathbf{r}_2, t)|^2 \rangle_t}}.$$

We distinguish between two types of (in)coherence:

- **Longitudinal coherence** refers to the variations in the normalized cross-correlation function $\gamma(\mathbf{r}_1, \mathbf{r}_1, \tau)$ with time delay τ . The longitudinal coherence length $\ell = c\tau_{\text{coh}} \approx c/\delta\nu$ is Fourier related to the spectral width of the optical field. A large longitudinal coherence length, corresponding to a small spectral width, yields a high visibility of Young's interference over many fringes. A short longitudinal coherence length, on the other hand, requires sources with a broad optical spectrum. This type of source is used when a high longitudinal spatial resolution is needed. It is used in **Optical Coherence Tomography (OCT)**, which records the interference of back-reflected light of a sample positioned in one arm of a Michelson interferometer with light from the other arm.
- **Transverse coherence** refers to the variations in the normalized cross-correlation function $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ with the position difference $\mathbf{r}_1 - \mathbf{r}_2$ in the transverse direction. As such, it can only be separated from the longitudinal coherence for a paraxial source of limited spectral width. The transverse or spatial coherence determines the visibility of the central fringes in Young's experiment. Even if the double slit is illuminated with quasi-monochromatic light, the interference will disappear if the **transverse coherence length** of the incident light is smaller than the slit separation, such that the optical fields at the two slits are on average uncorrelated. This situation occurs when the illumination is over too wide an angular range (see next section).

4.3 Spatial coherence & Van Cittert-Zernike theorem

Temporal or longitudinal coherence is a relatively straightforward concept, being Fourier related to the spectrum of the optical source. Spatial coherence is more complicated. A key idea in the theory of spatial coherence is the notion that the correlation function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle E^*(\mathbf{r}_1, t)E(\mathbf{r}_2, t + \tau) \rangle_t$ propagates like the optical fields. Propagation thus produces partial coherence, even in sources that originally had 'no spatial coherence' at all, i.e., for sources for which $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \propto \delta(\mathbf{r}_1 - \mathbf{r}_2)$, because optical diffraction spreads the field in the transverse direction. This notion is made quantitative in the **Van Cittert-Zernike theorem**, which states that "under illumination with a spatially-incoherent source, the spatial correlation function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ in a plane behind the source is Fourier-related to the intensity profile of the source". In mathematical terms:

$$(4.6) \quad \Gamma(\mathbf{r}_1, \mathbf{r}_2; \omega) \propto \int_{-\infty}^{\infty} |E_{\text{source}}(\mathbf{r})|^2 \exp(i2\pi\mathbf{r}(\mathbf{r}_1 - \mathbf{r}_2)/(\lambda L)) d\mathbf{r}.$$

for paraxial propagation over a distance L behind a spatially incoherent source $|E_{\text{source}}(\mathbf{r})|^2$. Two examples hereof are the incoherent illumination of a slit and of a circular aperture. In both cases the paraxial intensity profile in a plane at some distance L between the aperture is uniform on account of the incoherent illumination. The transverse coherence length in this plane is

$$(4.7) \quad \Delta x = (1.22 \times) \frac{\lambda}{\theta},$$

where the opening angle $\theta = L/D$ with D the diameter of the slit or aperture, and where the factor $(1.22 \times)$ applies only to the circular aperture. The spectral cross-correlation function $\Gamma(\mathbf{r}_1, \mathbf{r}_2; \omega)$ has dropped from its central maximum to its first zero at a distance Δx from the central axis.

Chapters 9 and 10 of ref. [BRO03] contain several intriguing examples of the importance of coherence. One of these is a discussion of the influence of atmospheric turbulence on the spatial coherence of starlight on earth. This influence can be characterized by the so-called Fried parameter $\mathbf{r}_0$, which is typically as small as 10 cm (see § 9.5 of ref. [BRO03]). Images produced by telescopes with larger apertures are generally not diffraction-limited but blurred by atmospheric turbulence ('seeing'). Two tricks to avoid this blurring have recently been developed. **Adaptive optics** improves the image by adjusting the shape of a deformable mirror on a millisecond timescale to counter the random atmospheric variations. **Aperture synthesis** improves the image by combining signals from several telescopes in a clever (coherent) way. This trick is commonly used in radio astronomy, where the large λ would require unrealistically large aperture diameters D for sufficient angular resolution.

4.4 Chaotic light versus laser light

Suppose we have a black box that contains either a lamp or a laser. How can we distinguish between the two? The simple answer could be: light from a lamp is generally spatially diffuse and spectrally broadband, whereas a laser generally emits a spatially coherent optical beam with limited spectral width. But the laser spectrum can also be wide, as is the case for pulsed lasers, and the properties of both sources can be modified anyway by (i) spatial filtering (with apertures and lenses), (ii) spectral filtering, and (iii) attenuating the laser intensity to match that of the filtered lamp. When we have thus made the cross-correlation function $\Gamma^{(1)}(\mathbf{r}_1, \mathbf{r}_2, \tau)$ of the laser and lamp identical, they will indeed be indistinguishable in any experiment that records only (time-averaged) intensity patterns.

We can, however, distinguish lamp light from laser light by recording the fluctuations in their output intensity $I(\mathbf{r}, t) \propto |E(\mathbf{r}, t)|^2$. These fluctuations are

described by the **normalized intensity correlation function**

$$(4.8) \quad \gamma^{(2)}(\tau) \equiv \frac{\langle I(t)I(t+\tau) \rangle_t}{\langle I(t)^2 \rangle_t}.$$

The intensity of laser light is constant ($\gamma^{(2)}(\tau) = 1$ for any τ), being stabilized by optical saturation of the laser gain medium (a nonlinear optical process). The intensity of the lamp fluctuates on the same timescale as its phase does ($\gamma^{(2)}(\tau) \approx 2$ for $\tau \ll 1/\Delta\omega$, $\Delta\omega$ being the spectral width of the light). This difference between laser light and lamp light is one of the key concepts in quantum optics; it is discussed briefly in §10.12-10.14 of ref. [BRO03] and more extensively in Chapters 6-8 of this syllabus.

Chapter 5

Optical systems

This material is based on chapters 11 and 12 of the book *Modern Classical Optics* [BRO03]. It introduces the optical étendue, as a measure for the combined spatial and angular spread of optical radiation and the light-gathering properties of optical systems, and discusses the Abbe limit of imaging and its consequences.

5.1 Etendue & number of transverse modes

The propagation of light through optical systems depends on the ‘spatial and angular spread’ of the radiation. These can be quantified by the **optical étendue** or **geometric extent**, which is defined as the product of the emitting area $dS_{\perp}$, in the direction perpendicular to the optical axis, times its solid angle $d\Omega$ times the refractive index squared n^2 . For emission in a cone with semi opening angle Θ inside a medium with refractive index n , we define the **numerical aperture** $NA \equiv n \sin \Theta$. The effective solid angle of this cone is easily calculated to be $\Omega_{\text{eff}} \approx \pi NA^2$ for a **Lambertian source**, being a source for which the emitted power $dP \propto dS_{\perp} \propto \cos \theta$, to account for the reduced effective source size under non-zero viewing angle. The **normalized étendue**

$$(5.1) \quad N = \frac{dS d\Omega_{\text{eff}}}{\lambda_0^2} = \frac{dS \pi (\sin \Theta)^2}{\lambda^2},$$

quantifies the number of transverse optical modes (at a single polarization; $\lambda = \lambda_0/n$).

Some concepts: The **radiance** $B = dP/(dS_{\perp} d\Omega_{\text{eff}})$ is the optical power per unit area per unit of “effective” opening angle. The normalized radiance measures the power per transverse mode dP/dN . The **spectral radiance** or **spectral brightness** is the radiance per spectral bandwidth.

The étendue is such a powerful concept because it is invariant under imaging, where an increase in image size is always accompanied by a reduction in the opening angle of the illumination, and vice versa. Light just cannot be concentrated in a smaller phase space volume of fewer transverse modes, a result that also follows from thermodynamic (entropy) arguments. The only way to reduce the étendue in a linear optical system is to remove the unwanted modes at the expense of a power reduction. Only in nonlinear optical systems, such as optically-pumped lasers, can the optical energy out of many modes be concentrated into fewer modes.

In ref. [BRO03], Brooker compares the spectral radiance of various classical sources, by linking the average number of photons per mode to the effective radiative temperature of the source. The bottom line is that (i) incandescent (=thermal) sources emit broadband spectra with effective temperatures up to 3000 K, (ii) gas discharge lamps emit line spectra with effective temperatures up to 6000 K ($\approx$ temperature of sun), and even (iii) light-emitting diodes (LED) produce $\ll$ one photon per transverse mode per second per Hz spectral bandwidth. Only lasers emit light with a much larger normalized spectral radiance (typically 10^8 photons per mode). This is mainly a matter of concentration: the absolute power is generally still small (milliWatts to Watts) but concentrate in a single transverse mode and a very narrow optical spectrum.

The optical étendue is crucial in the description of the the light gathering performance of optical instruments. Brooker argues why optical interferometric instruments that are based on amplitude splitting, such as the Michelson interferometer, are generally much more efficient in light gathering than instruments that split the optical phase front, such as Young's double slit or the grating spectrometer. The reason being that the amplitude splitting devices can produce interference over a full image, even if the radiations is spatially incoherent. Phase-front splitting devices require initial spatial filtering to create sufficient spatial coherence; they work fine only with a single transverse mode in the direction perpendicular to the slits or grating lines. With modern CCD imaging devices, which allow for single-shot inspection of the full optical spectrum, the spatial disadvantage of phase-front splitting devices is, however, more than compensated by the efficient multi-channel detection.

5.2 Abbe limit of resolution

The optimum resolution of an ideal aberration-free imaging system is set by Abbe's **diffraction limit**

$$(5.2) \quad \Delta x = 1.22 \frac{\lambda_0}{2NA},$$

where $NA = n \sin \Theta$ is the numerical aperture of the first collection lens, with (half-width or semi-) opening angle Θ , and n is the refractive index of a possible immersion medium. The diffraction limit basically combines the equation for angular diffraction $\Delta\theta = (1.22 \times) \lambda / D$ with the equation $\Delta x = f \Delta\theta$. It can also be interpreted as a Fourier relation, by noting that “the optical amplitude in the focal plane of a lens is the Fourier transform of the amplitude profile in the object plane”. The opening angle of the lens now limits the maximum Fourier component k_{trans} that can be properly imaged through the system. Brooker states that Abbe’s diffraction limit can only be reached under wide-angle illumination, produced with a so-called **condenser** lens.

The optical resolution under coherent illumination can be conveniently described by its **point-spread function**

$$(5.3) \quad h(x, y) \propto \int_{-\infty}^{\infty} H(k_x, k_y) \exp[-i(k_x x + k_y y)] dk_x dk_y,$$

where $H(k_x, k_y)$ is the transmission function of the lens system, expressed in terms of transverse wavevectors. The point-spread function describes the amplitude spread of the image of an ideal point source. Under incoherent illumination, this spread is instead determined by the associated intensity profile $|h(x, y)|^2$. Its Fourier transform, known as the **optical transfer function** $OTF(k_x, k_y)$, describes how the optical system filters space frequencies under incoherent illumination.

5.3 Tricks in microscopy

Brooker describes several tricks to enhance the image contrast in microscopy. Some techniques employ phase plates positioned in the focal plane of the imaging lens to convert (invisible) phase variations in the light transmitted by a sample into (visible) amplitude variations. Brooker[BRO03] mentions three examples: **phase-contrast microscopy** (phase shift π of light in central region), **dark-ground illumination** (light in central region is blocked), and **Schlieren** (light in half plane is blocked). In another technique, called **dark-field illumination**, the sample is illuminated only at angles that lie beyond the maximum collection angle of the lens such that only scattered light contributes to the image.

As a final and very important trick to improve the spatial resolution we mention the scanning **confocal microscope**, described in section 16.8-16.10 of ref.[BRO03]. This microscope combines sharp imaging with sharp localized illumination. The confocal microscope has a somewhat higher resolution than ordinary microscopes; typically a factor $\sqrt{2}$ when illumination and imaging stages have the same resolution. It also has a finite depth of focus that allows preferential imaging of thin sheets of

material within a given volume. Finally, the addition of pinholes in the illumination and imaging stage, combined with sample scanning, can remove unwanted (stray) light.

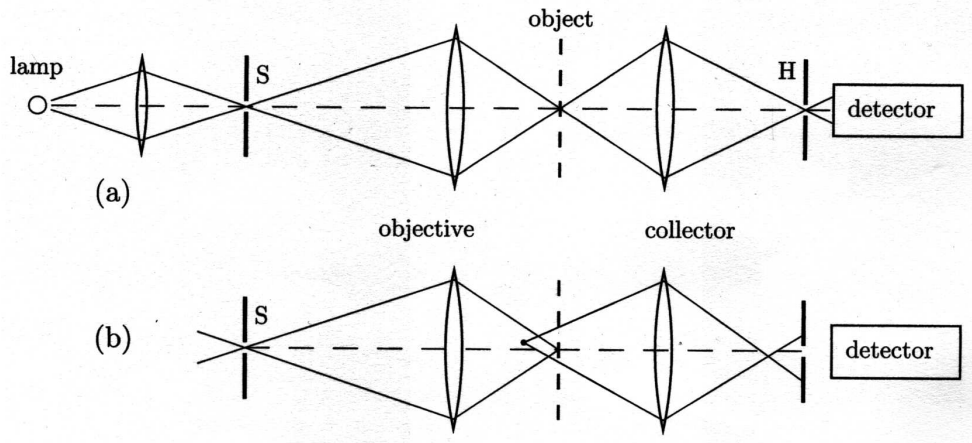


Figure 5.1: (a) The illumination in a confocal microscope, originating from a point-like source S , is limited to a small part of the object only. A confocal image is constructed by scanning the object while monitoring the transmission/fluorescence behind a second aperture H . (b) The “double focusing” configuration leads to a slight increase in spatial resolution and a dramatic reduction of stray light from out-of-focus objects. (Fig. 6.4 of [BRO03])

5.4 Optical aberrations in imaging

Imaging systems are often not as ideal as one would like them to be due to all kinds of **imaging aberrations**. Although modern lens design is a highly specialized job, which Brooker describes as “a form of computer-aided trial-and-error”, it is still nice to know a few fundamental aspects of optical aberrations. Complete textbooks have been written on the topic [MAH98] and applications are numerous, especially for companies like ASML.

In aberration theory, one always compares path lengths of rays in the optical system under study with path lengths in the ideal non-aberrated imaging system. The path length difference ds is obviously a function of the chosen ray. It is typically specified as a function of three variables: (i) the off-axis displacement r of the ray on the image lens (the so-called pupil plane), (ii) the off-axis displacement h of the object - or equivalently the off-axis displacement h' of the image - and (iii) the azimuthal angle φ between the points in the (2-dimensional) pupil plane and

object (or image) plane. Symmetry arguments show that only five so-called **Seidel aberrations** contribute in the lowest-order non-trivial Taylor expansion [MAH98], which reads:

$$(5.4) \quad ds = a_{40}r^4 + a_{31}r^3h' \cos \varphi + a_{22}r^2(h')^2(\cos \varphi)^2 + a_{20}r^2(h')^2 + a_{11}r(h')^3 \cos \varphi.$$

These five terms are denoted as follows:

- **Spherical aberration** $a_{40}r^4$ describes the modified focussing of rays that don't pass through the center of the lens; it is the only non-zero Seidel aberration for an on-axis object point ($h = h' = 0$).
- **Coma** $a_{31}r^3h' \cos \varphi$ describes a variation in the imaging for an off-axis point. After integration over the pupil plane (r and φ) it results in a cone-shaped image of a point-like object.
- **Astigmatism** $a_{22}r^2(h')^2(\cos \varphi)^2$ describes the second important variation in the imaging for an off-axis point. It results in a displacement of the focus of rays coming from different transverse directions. More specifically, rays that lie in the plane spanned by the optical axis and the object point produce the so-called tangential line image, whereas rays that lie in the orthogonal plane produces the so-called sagittal (or radial) line image *in a different longitudinal plane*. The circle of least confusion is visible in between these two imaging planes.
- **Field curvature** $a_{20}r^2(h')^2$ results in a h' -dependent longitudinal displacement of the image point. It can effectively be removed by observing the image in a curved instead of a planar reference plane.
- **Distortion** $a_{11}r(h')^3 \cos \varphi$ results in a h' -dependent transverse displacement of the image point, without blurring the focus. It can hence also be removed by re-scaling the image plane.
- Extra aberration: **Chromatic aberration** becomes important at increased optical bandwidth, when different colors might be focused in different points.

5.5 Spherical aberrations

When imaging an on-axis object or focussing a laser beam into a tight spot, the **Spherical aberration** is the only relevant Seidel aberration. Hence, we'll discuss some properties of this dominant aberration:

- Even perfect spherical lenses produce spherical aberrations on account of Snell’s law and the lowest-order non-trivial term in the Taylor expansion $\sin \theta \approx \theta - \theta^3/6 + \dots$. The magnitude of the spherical aberrations can be quantified by the coefficient a_s (with dimension [m]) in the Taylor expansion of the extra optical path length[MAH98]

$$(5.5) \quad ds = a_s \theta^4,$$

experienced by off-axis optical rays as a function of the ray angle at the image. This deviation of the phase front from a spherical wave results in a transverse displacement by $\Delta x = 4|a_s \theta^3|$ of the off-axis ray in the image plane. The resulting “**blur circle**” around the diffraction-limited image can be reduced somewhat by shifting the reference image plane away from its paraxial position.

- The spherical aberrations produced by a single-lens imaging system are always negative ($a_s < 0$) if both object and image are real, i.e., rays focused by the outer parts of the lens always cross the optical axis somewhat closer to the lens than the paraxial rays. As a result, the image observed under coherent illumination changes from a “bull’s eye” image, exhibiting interference rings, in a plane close to the lens, into a “powder box” (fluffy) image further away from the lens.

The spherical aberrations introduced by a single spherical lens depends on (i) the shape of the lens and (ii) the collimation of the rays, i.e., the magnification from object to image. At large magnification, i.e., for far-away objects, close to optimum imaging is obtained for a plano-convex lens oriented with its plano-side towards the focus. This geometry yields $a_s \approx -0.27f$ and $\Delta x \approx 1.1f\theta^3$ for $n = 1.5$. Shifting the reference plane somewhat, the latter equation reduces to a minimum blur circle of

$$(5.6) \quad \Delta x \approx 0.54f\theta^3.$$

(see [OFR]), to be compared with the diffraction limit $\Delta x_{\text{diff}} \approx 0.61\lambda/\theta$.

Equation (5.6) describes the practical limitations of standard spherical optical in imaging. It shows that spherical aberration is already a serious problem for imaging at $NA \geq 0.1$, as the minimum blur circle calculated from Eq. (5.6) already exceeds the diffraction limit of about $4 \mu\text{m}$ @ $\lambda \approx 650 \text{ nm}$ for focal lengths as small as 8 mm. Imaging at $NA > 0.1$ therefore always requires a combination of at least two spherical lenses, generally including a meniscus lens, or specially aspherical lenses.

- Even the insertion of a plan-parallel plate in a converging optical beam introduces spherical aberrations if the refractive index of this medium $n \neq 1$.

This aberration is positive at $a_s = t.(n^2 - 1)/(8n^3)$ for a plate of thickness t . Although smaller than the spherical aberration that are typically induced by lenses, this aberration can be crucial for imaging at larger NA . Hence, even the presence of a plan-parallel plate should be included in the lens design. Lenses that are designed for imaging without a plate can still be used reasonable well for imaging through a plate and vice versa if the plate is not too thick and if $NA \leq 0.3 - 0.4$. Stallinga[STA05] has shown analytically how the plate-induced spherical aberrations can be largely compensated for by operating the imaging lens at a different magnification, i.e., by using a divergent input beam instead of a collimated one.

- High-quality large NA (aspheric) lenses or lens systems are always designed for specific imaging geometries. Metallurgic objectives are designed for imaging in free-space; biological objectives are designed for imaging through glass cover plates (typical thickness 150-200 μm); immersion objectives are designed for imaging in a liquid with a specific refractive index. The imaging quality can be seriously degraded when one deviates from the design geometry.

5.6 Tight focusing and vector diffraction

Tight focusing is incompatible with a scalar description of the optical field. This is a consequence of the Maxwell equations, which a.o. contains the equation $\nabla \cdot \mathbf{E} = 0$. Consequences of tight focussing are:

- The introduction of a field component in the direction of propagation $E_{\parallel}/E_{\perp} \approx \theta$. This parallel field has been used to accelerated electrons and ions to very high energies (MeV - GeV) with intense and tightly-focussed femtosecond optical pulses.
- If the optical polarization is uniform/pure in a collimated beam, it will not be uniform anymore when this beam is focused by a lens and vice versa. The orthogonal field component introduced by focusing is $E_y/E_x \propto \theta^2$ for x polarized light. This component is absent along the x and y axis and exists only on the four diagonal in the xy plane.[ERI94]
- The size of the focus in the direction parallel to the optical polarization is somewhat larger than the size in the perpendicular direction, at $w_{\parallel}/w_{\perp} \approx 1 + \theta^2/2$. I like to attribute this difference to a difference in the Fourier decomposition $\mathbf{E}_{\perp}(\mathbf{k})$ of the field. The angular distribution of the TE-polarized component is typically a bit more compact than that of the TM-component

because $E_{\perp,TE}/E_{\perp,TM} \propto \cos \theta$ due to projection. This makes the size of the focus in the direction of the polarization somewhat larger.

Chapter 6

Semi-classical photon statistics

Experiments that only measure the average optical power, either directly or behind linear optical devices such as interferometers or spectral-filters, can always be described by classical theory and do not require a quantum description. True quantum behavior is observable only in correlation experiments aimed at measuring the intensity fluctuations or **photon statistics** of the optical field. The next three chapters describe such correlation experiments from three points of view: (i) a semi-classical description, (ii) a single-mode quantum description, (iii) a continuous-mode quantum description that includes the full time dynamics of the optical field. The semi-classical description presented in this chapter is based on Chapters 5 and 6 of the book *Quantum Optics* by M. Fox [FOX06].

6.1 Fluctuations in the photon flux

Fluctuations in the optical intensity can be measured either with very sensitive photodiodes, which basically record the incident optical power $P(t)$ as a function of time, or with photon counters, which produce a discrete “click” for every detected photon. Our discussion will be centered around the photon statistics recorded in the later experiment. We consider a photon counter with quantum efficiency η = average number of electronic pulses (clicks) divided by the average number of incident photons. Under illumination with weak light at an average power $\bar{P}$, the average count rate of the photon detector is

$$(6.1) \quad \bar{R} = \eta \bar{P} / (\hbar\omega),$$

where $\hbar\omega$ is the energy per photon. We will first consider the ideal detector with $\eta = 1$.

Fox correctly points out that it is unclear whether the temporal fluctuations in the measured count rate should be attributed to (1) the statistical nature of the detection process, or (2) the intrinsic photon statistics of the light beam. Most results obtained in photon counting experiments, in particular those with coherent or thermal light, can in fact be explained by a **semi-classical model**, where the incident optical field is treated classically and where the quantum aspects are limited to the atoms/molecules in the detector. Only experiments that yield **sub-Poisson photon statistics** or **photon anti-bunching** require a quantum description of the optical field.

On first sight, it seems reasonable to discuss photon statistics in terms of the temporal variations in the detected photon flux $R(t)$. However, due to the discrete nature of the photons, the detected signal will consist of sharp delta-like spikes with a time structure that merely contains information on the detector speed. Hence, we will instead consider the number of detection events

$$(6.2) \quad n \equiv \int_0^T R(t) dt,$$

in a fixed time window T and in particular its probability distribution $P(n) = P_n$. The average number of counts in this time interval is $\bar{n} \equiv \sum n P_n$; the variance in the count number is $\Delta n^2 \equiv \sum (n - \bar{n})^2 P_n$. We distinguish three different cases (see §5.3-5.6 of Fox):

- **Super-Poissonian** statistics ($\Delta n > \sqrt{\bar{n}}$), with $\Delta n^2 = \bar{n}^2 + \bar{n}$ for thermal light,
- **Poissonian** statistics ($\Delta n = \sqrt{\bar{n}}$) for coherent light, and
- **Sub-Poissonian** statistics ($\Delta n < \sqrt{\bar{n}}$) for non-classical light.

The semi-classical theory of light describes the optical field classically, in terms of its classical optical field $E(t)$ and intensity $I(t) \propto |E(t)|^2$, while the detection is treated as a discrete quantum process. The only quantum-mechanical input in this theory is the assumption that the probability to generated one additional photoelectron in a short time interval Δt is proportional to the (average) intensity, i.e., that $\Delta P \propto I(t) \Delta t$.

The semi-classical theory is correct only for light with Poissonian and super-Poissonian statistics. Coherent light, with its associated Poissonian statistics, can be modeled as classical light with a constant intensity $I(t) = I_0$. Thermal light, on the other hand, corresponds to a classical field with wild intensity fluctuations described by an exponential probability distribution $P(I) \propto \exp(-I/\bar{I})$ and an associated Gaussian probability distribution of its complex field $P(E) \propto \exp(-|E|^2/|\bar{E}|^2)$.

The Poissonian statistics of the photons detected in coherent light arises from the random picking of photons. The super-Poissonian photon statistics of thermal light results from its intrinsic intensity fluctuations $\Delta I^2 = \bar{I}^2$, which, in combination with the random picking of photons, results in $\Delta n^2 = \bar{n}^2 + \bar{n}$. The corresponding photon number distributions P_n are depicted in Fig. 6.1.

Sub-Poissonian photon statistics cannot be described in semi-classical terms. It requires a more regular stream of detection events as if the photons “try to avoid each other”. This property is generally referred to as “photon anti-bunching”, to be compared with “photon bunching” in thermal light and the “uncorrelated photons” in coherent light.

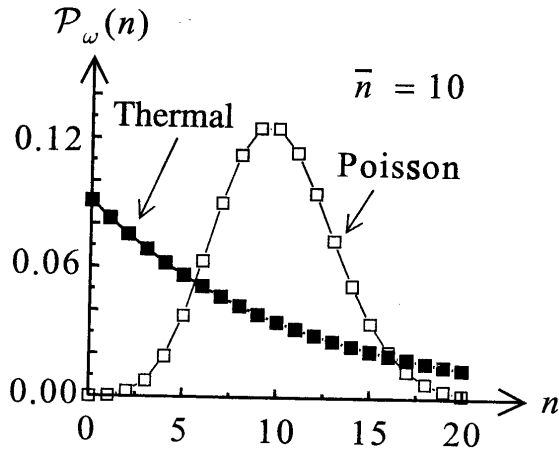


Figure 6.1: Comparison of the photon statistics for a single mode of a thermal source with average photon number $\bar{n} = 10$ and a coherent source with the same $\bar{n}$ (Poisson distribution) (Fig. 5.5 of [FOX06]).

Let me finish this section with a word of warning. The statistical analysis presented above is only valid if the detected optical intensity is concentrated in a “single mode of the optical field”. This requirement refers to the optical polarization as well as the spatial and spectral degrees of freedom of the field. More specifically, we only considered a single optical polarization of an optical field in a fixed spatial mode, like the field in a single-mode optical fiber. Furthermore the inspection time T should be smaller than the optical coherence time (= inverse optical bandwidth). The reason for this requirement is simple: as the optical fields in different modes are generally uncorrelated, the photon statistics of the combined intensity of many modes tends to be close to Poissonian *for any type of light* on account of the central limit theorem.

6.2 Photon statistics changes with losses

The deteriorating effect of losses on the detected photon statistics is described in §5.8.2 and 5.10 of the book of Fox.[FOX06] This effect is essentially described by the

idea that any loss $\eta \equiv P_{\text{detected}}/P_{\text{input}}$, either in the optics or in the detection process, can be described by the random removal of a fraction $1 - \eta$ of the photons. Loss obviously reduces the average counts by $\bar{N} = \eta\bar{n}$, where n and N are the number of incident and detected photons within a fixed time interval T , respectively. The random-picking statistics also yields $\Delta N^2 = \eta^2 \Delta n^2 + \eta(1 - \eta)\bar{n}$, which can be rewritten as

$$(6.3) \quad \frac{\Delta N^2}{\bar{N}} = \eta \frac{\Delta n^2}{\bar{n}} + (1 - \eta).$$

Equation (7.13) shows that loss doesn't modify the generic statistical properties of either thermal light ($\Delta N^2 = \bar{N}^2 + \bar{N}$) or coherent light ($\Delta N^2 = \bar{N}$). It does, however, degrade the sub-Poissonian statistics of non-classical light, making it more Poissonian, as demonstrated by the extreme case $\Delta N^2 = (1 - \eta)\bar{N}$ for the case of fully anti-bunched input $\Delta n^2 = 0$.

The ratio $F = \Delta N^2/\bar{N}$ is called the **Fano factor**. The Fano factor described how much the photon statistics deviates from Poissonian statistics, where $F > 1$ correspond to super-Poissonian statistics and $F < 1$ corresponds to sub-Poissonian statistics. Eq. (7.13) shows that losses will always pull the Fano factor towards its preferred value of 1 via $(F_{\text{out}} - 1) = \eta(F_{\text{in}} - 1)$. For coherent input, the relation $F_{\text{out}} = F_{\text{in}} = 1$ is straightforward. For thermal input, the relation $F_{\text{int}} = \bar{n} + 1$ changes into a similar relation $F_{\text{out}} = \bar{N} + 1$. For fully anti-bunched input, we obtain $F_{\text{out}} = 1 - \eta$. All these results can still be explained by semi-classically, if we introduce the concept of **vacuum fluctuations** to the semi-classical theory. The loss-induced transformation from sub-Poissonian to Poissonian photon statistics allows for a simple intuitive interpretation. When the optical field is interpreted as a regular stream of discrete photons, the “random removal of photons” will naturally introduce noise to the system. In order to quantify this effect in a semi-classical theory, one generally introduces **vacuum fluctuations** as an unavoidable by-product of losses. These vacuum fluctuations, which give rise to the $(1 - \eta)$ term in Eq. (7.13), unavoidably leak in through the second port of the beam splitter or any other component that models the optical loss. I consider vacuum fluctuations to be the “classical interpretation” of the commutation relations of the field operators that play an important role in the full quantum-mechanical description of the optical field.

6.3 Shot noise

The equivalent of Poissonian statistics in a photon stream is shot noise in the detected photo current. The optical **equivalent noise power** (NEP) P_{shot} in a co-

herent optical beam of average power P_{in} is

$$(6.4) \quad P_{\text{shot}} = \sqrt{2\hbar\omega\eta P_{\text{in}}\Delta f} = \frac{\hbar\omega}{e} \sqrt{2eI\Delta f},$$

where Δf is the electronic detection bandwidth, η is the detection efficiency and I is the detected photo current. This square-root expression for the optical noise power is the equivalent of the relation $\Delta n = \sqrt{\bar{n}}$ for coherent light. Experimentally, one considers the scaling $P_{\text{noise}} \propto \sqrt{P_{\text{in}}}$ as a proof of the dominance of shot noise, as classical noise scales as $P_{\text{noise}} \propto P_{\text{in}}$ whereas (electronic) detector noise should be independent of P_{in} . Likewise, the observation of a noise power below the shot-noise limit is an experimental proof of the non-classical nature of the source. The later experiments are generally very difficult as any loss will effectively introduce quantum noise and make the signal “more classical”. For completeness, we note that Fox quantifies shot noise in a different way, using the **electronic noise power** $P_{\text{electronic}} \equiv R_L \langle \Delta I^2 \rangle = 2eR_L \bar{I} \Delta f$ dissipated in a load resistor R_L , instead of the equivalent optical noise power.

6.4 Hanbury Brown & Twiss experiments

One might think that intensity fluctuations in a beam are most easily measured by simply recording the beam’s intensity as a function of time, or by performing photon counting experiment in a fixed time window T to determine the probabilities P_n . This is not the case; it is easier to experimentally split an incident beam in two parts and correlate the two measured beam intensities. The first experiments of this sort were performed by Hanbury Brown and Twiss in the mid 1950’s [HAN56, TWI56]. Their experiment, and variations on its theme, played such an important role in the development of quantum optics that they are discussed in several chapters of this syllabus and also in the book of Fox [FOX06] in § 6.2 - 6.7 and § 8.5.

We consider the intensity correlation experiment of Hanbury Brown and Twiss, presented in 6.2. In the semiclassical description, the two beam intensities are equal and can be described as $I_1(t) = I_2(t) = \bar{I} + \Delta I(t)$. We characterize the strength and dynamics of the intensity fluctuations by the second-order correlation function of the optical field

$$(6.5) \quad g^{(2)}(\tau) \equiv \frac{\langle I(t)I(t+\tau) \rangle}{\langle I(t) \rangle \langle I(t+\tau) \rangle} = 1 + \frac{\langle \Delta I(t)\Delta I(t+\tau) \rangle}{\langle I(t) \rangle^2},$$

where $\langle \rangle$ denotes averaging over t and assuming stationary light (= statistical properties do not depend on the absolute time t). The **second-order coherence function** $g^{(2)}(\tau)$ is called second-order because it is second order in the optical field E .

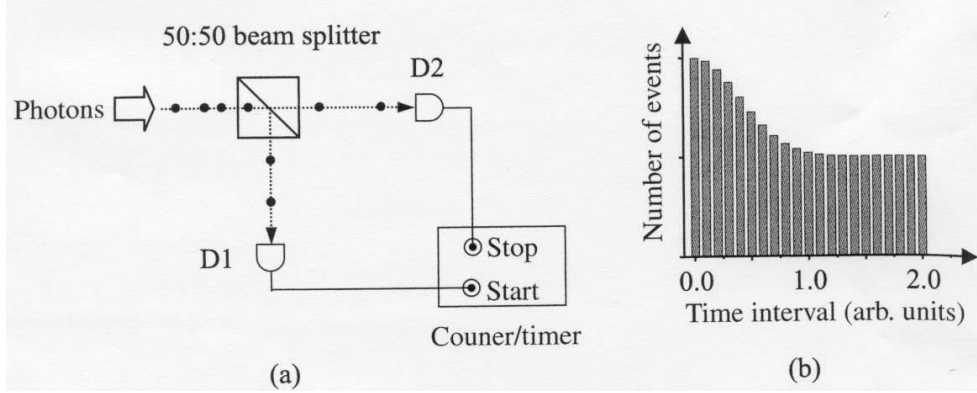


Figure 6.2: Photon correlations can be measured with a setup developed by Hanbury Brown and Twiss. (a) An incident beam is split and directed to two photon counters. A start-stop timer/counter records the statistics of the arrival times at detector D1 and D2 versus the time difference of arrival. (b) A typical result, demonstrating photon bunching of thermal/chaotic light (Fig. 6.5 of [FOX06]).

A similar first-order coherence function is defined as

$$(6.6) \quad g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau) \rangle}{\langle |E(t)|^2 \rangle},$$

where $E(t)$ is the complex (= positive frequency part of the) optical field. This first-order coherence function is Fourier related to the optical spectrum (see also Sec. 8.2).

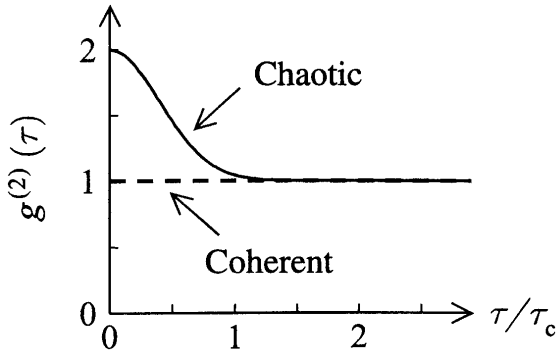


Figure 6.3: Second-order correlation function $g^{(2)}(\tau)$ for thermal/chaotic light and coherent light (Fig. 6.4 of [FOX06]).

The second-order correlation function of semi-classical light always peaks at $\tau = 0$, i.e., $g^{(2)}(0) \geq g^{(2)}(\tau)$ for $\tau \neq 0$ and decays to $g^{(2)}(\tau) = 1$ for $\tau \rightarrow \infty$ on account of the finite memory time of practically any source. However, values $g^{(2)}(\tau) < 1$ are possible for $\tau \neq 0$ if the classical $I(t)$ fluctuates within a limited range of frequencies. For instance, classical light with $I(t) = I_0 [1 + A \sin(\omega_m t)]$ yields $g^{(2)}(\tau) = 1 + \frac{1}{2}A^2 \cos(\omega_m \tau)$ thus oscillating between values of 1.5 and 0.5.

In a quantum-mechanical description of the Hanbury Brown & Twiss experiment, the second-order correlation function is defined as

$$(6.7) \quad g^{(2)}(\tau) \equiv \frac{\langle n_1(t)n_2(t+\tau) \rangle}{\langle n_1(t) \rangle \langle n_2(t+\tau) \rangle},$$

where $n_i(t)$ is the number of counts registered on detector i around time t and where the numerator refers to the (almost) simultaneous detection of one photon at detector 1 and another photon at detector 2. In other words, $g^{(2)}(\tau)$ is proportional to the **conditional probability** of detecting a second photon at time $t = \tau$, given that we detected the first photon at $t = 0$. Or in equation form, $g^{(2)}(\tau) = P(t + \tau|t)/P$, where $P(t + \tau|t)$ is the conditional probability for detection of a (second) photon at time $t + \tau$ after detection of a (first) photon at time t and where $P = \lim_{\tau \rightarrow \infty} P(t + \tau|t)$. Based on the above definition of $g^{(2)}$ one can make the following classification (see [FOX06] § 6.4-6.5):

- bunched light: $g^{(2)}(0) > 1$ (with $g^{(2)} = 2$ for thermal light)
- coherent light: $g^{(2)}(0) = 1$
- anti-bunched light: $g^{(2)}(0) < 1$

This behavior is demonstrated in Figs. 6.3 and 6.4.

Finally, a “word of warning” in relation to the requirement of single mode detection in the experiment of Hanbury Brown and Twiss. The contrast $g^{(2)}(0) - 1$ can be strongly compromised if the observation is not limited to a single spatial mode. Likewise, the temporal resolution should be better than the optical coherence time (= inverse spectral bandwidth) in order to compare photons within the same $(\Delta t, \Delta \nu)$ segment of phase space. The original experiment of Hanbury Brown and Twiss was performed on spatially-filtered light within a single spectral line of a Mercury lamp (=thermal source). The spatial resolution was reasonably OK, but the detector response was at least a factor 10 slower than the inverse width of the Doppler-broadened line. The observed photon bunching was thus limited to $g^{(2)}(0) = 1 + 1/M \approx 1.03$ instead of the theoretical maximum of 2, M being the number of detected spectral and spatial modes. Instead of trying to improve on this value, Hanbury Brown and Twiss made a more drastic move by switching from a Mercury lamp to star light in consecutive experiments. These new experiments allowed them to measure the angular diameter of tens of stars, using an intensity-correlation technique that is per definition insensitive to the path-length variations that generally frustrate the competing first-order interference experiments.

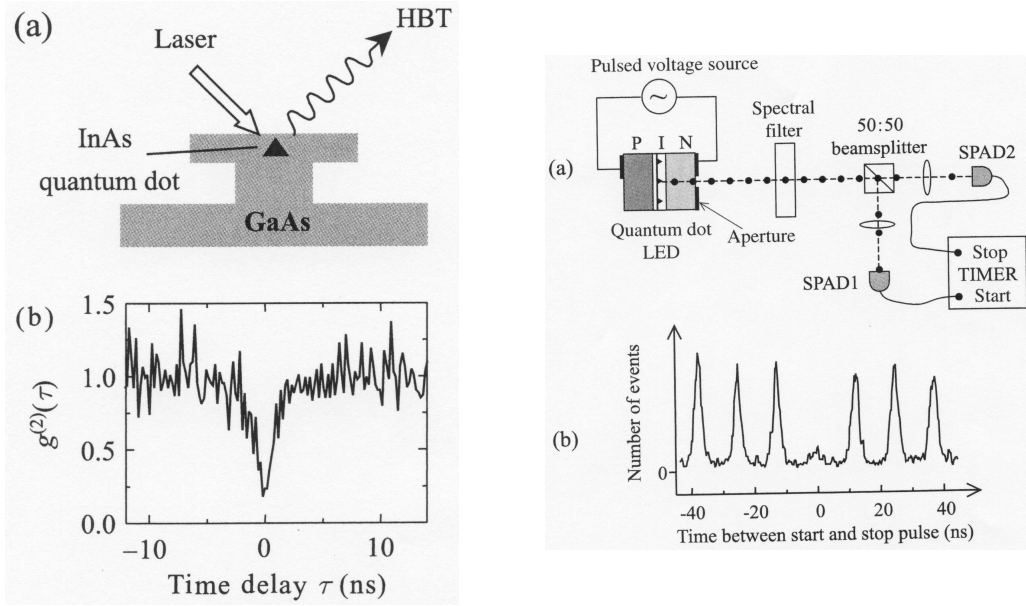


Figure 6.4: Two key experiments that demonstrated the existence of photon anti-bunching in the emission of a single-photon source. (left) Photon correlations observed under continuous-wave excitation of a single InAs quantum-dot emitter. (right) Photon correlations observed under pulsed excitation of a similar single-photon source at a repetition time of 13 ns, showing the absence of double-click events from single optical pulses (Figs. 6.11 and 6.13 of [FOX06]).

Chapter 7

Single-mode Optics

The next two chapters introduce the more mathematical formulation of quantum optics, which is based on the second quantization of the optical field and the introduction of photon creation and annihilation operators. The general part is based on Chapters 7 and 8 of Fox's book. The more formal part (Sec. 7.2) is based on the excellent book *The Quantum Theory of Light* by R. Loudon [LOU03]. In this chapter we will discuss the quantum properties of a single discrete mode of the optical field, such as the optical field inside a high-finesse optical cavity. In the next chapter, we will instead analyze a continuum of modes. This important difference is emphasized in the book *The Quantum Theory of Light* [LOU03] by Loudon's chapter titles "Single-mode quantum optics" versus "Multi-mode and continuous-mode quantum optics".

7.1 Annihilation and creation operators

The quantization of the electro-magnetic field generally starts with its separation in discrete spatial modes. This is achieved by considering the field inside a closed rectangular box and applying periodic boundary conditions of the form $E(0, 0, 0) = E(L_x, 0, 0)$ to all boundaries. The discrete travelling-wave modes of this system are labeled by their wave vector $\mathbf{k} = (N_x \frac{2\pi}{L_x}, N_y \frac{2\pi}{L_y}, N_z \frac{2\pi}{L_z})$ and polarization $\mu = \{1, 2\}$. Next we separate the optical field in a positive and negative frequency component, as $E(t) \equiv E^+(t) + E^-(t)$ where $E^-(t) \equiv (E^+(t))^*$, and use the mode expansion

$$(7.1) \quad \hat{E}^+(\mathbf{r}, t) = \sum_{\mathbf{k}, \mu} \left(\frac{\hbar \omega_{\mathbf{k}}}{2\epsilon_0 V} \right)^{\frac{1}{2}} \mathbf{e}_{\mathbf{k}, \mu} \hat{a}_{\mathbf{k}, \mu} \exp(-i\omega t + i\mathbf{k} \cdot \mathbf{r}),$$

where $\mathbf{e}_{\mathbf{k},\mu}$ is the polarization direction. The hats indicate that both the electric field $\hat{E}^+(\mathbf{r}, t)$ and the modal amplitude $\hat{a}_{\mathbf{k},\mu}$ should actually be interpreted as quantum-mechanical operators. The factor $[\hbar\omega_{\mathbf{k}}/(2\epsilon_0 V)]^{\frac{1}{2}}$ is chosen such that $\hat{a}_{\mathbf{k},\mu}^\dagger$ and $\hat{a}_{\mathbf{k},\mu}$ are creation and annihilation operators that raise and lower the number of photons in the considered mode by exactly one photon.

The field energy contained in this box can be written in terms of the electromagnetic fields E and B as

$$(7.2) \quad \mathcal{H} = \frac{1}{2} \int dV (\epsilon_0 |\mathbf{E}|^2 + (1/\mu_0) |\mathbf{B}|^2).$$

Substitution of Eq. (7.1), and a similar equation for the H -field, into Eq. (7.2) yields the quantum Hamiltonian

$$(7.3) \quad \hat{\mathcal{H}} = \sum_{\mathbf{k},\mu} \hbar\omega_{\mathbf{k}} \left(\hat{a}_{\mathbf{k},\mu}^\dagger \hat{a}_{\mathbf{k},\mu} + \frac{1}{2} \right).$$

This result, in combination with the field operator commutation relation of Eq. (7.5), is called the **second quantization method**, or more appropriate the **occupation number representation** of the optical field.[DEB65] The **first quantization** in quantum mechanics attributes wave-like properties to a single particle via its probability wave function ψ . The second quantization does the opposite; it describes the properties of a field in which particles can be created or destroyed, thus attributing particle-like properties to the field. It does so by replacing the classical field variable by a quantum operator. In **quantum field theory**, the amplitude of the field becomes quantized and the quanta are identified with individual particles.

Quantum mechanics predicts that each mode contributes $\frac{1}{2}\hbar\omega_{\mathbf{k}}$ to the **vacuum energy** even if this mode is “not occupied”. As the time-averaged energy of the electric and magnetic fields are equal, this corresponds to a mean-square electric field strength of

$$(7.4) \quad \langle E_{\text{vac}}^2 \rangle = \frac{\hbar\omega}{2\epsilon_0 V},$$

per mode. The smaller the quantization volume V , the larger the mean-square field per mode. The divergence of the vacuum energy over the infinite sum of modes fortunately drops out of most theoretical expressions.

Very roughly speaking, the difference between the quantum and classical description of optical phenomena is that the former description uses quantum operators while the latter uses (complex) numbers. In the operator description, the ordering of the operators is of crucial importance and described by the **bosonic commutation relations**

$$(7.5) \quad [\hat{a}_{\mathbf{k},\mu}, \hat{a}_{\mathbf{k}',\mu'}^\dagger] \equiv \hat{a}_{\mathbf{k},\mu} \hat{a}_{\mathbf{k}',\mu'}^\dagger - \hat{a}_{\mathbf{k}',\mu'}^\dagger \hat{a}_{\mathbf{k},\mu} = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mu,\mu'}.$$

The polarization degree of freedom will be neglected from now onwards; operators of modes with orthogonal polarizations simply commute.

7.2 Quantum states of light

We introduce three important single-mode quantum states: (i) **number states**, (ii) **coherent states**, and (iii) **thermal light**:

(i) The **n-photon number state** $|n\rangle$, with energy $E_n = (n + \frac{1}{2})\hbar\omega$, can be created by repeated application of the raising operation

$$(7.6) \quad \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle,$$

where the factor $\sqrt{n+1}$ ensured normalization $\langle n|n\rangle = 1$, where the vacuum state is defined by the relation $\hat{a}|0\rangle = 0$, and where the mode labels $\mathbf{k}, \mu$ have been dropped for convenience. Different number states are orthogonal: $\langle i|j\rangle = \delta_{i,j}$. Number states are eigenstates of the photon number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ that measures the number of photons, as $\hat{n}|n\rangle = n|n\rangle$.

(ii) The **coherent state** $|\alpha\rangle$ is the most classical state of light. Coherent states are characterized by a single complex number α , which corresponds to the complex amplitude of the associated classical field. In quantum theory, this classical amplitude is surrounded by a “quantum probability cloud” (see Fig. 7.2). Coherent states can be generated from the vacuum by application of the coherent-state displacement operator $\hat{D}(\alpha)$ via

$$(7.7) \quad |\alpha\rangle \equiv \hat{D}(\alpha)|0\rangle \equiv \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Coherent states are generally not orthogonal, but obey the following relations

$$(7.8) \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad , \quad \langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha| \quad , \quad |\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha - \beta|^2)$$

The calculation of $\hat{a}^\dagger|\alpha\rangle$ is more difficult.

The probability of measuring n photons in the projection of coherent state $|\alpha\rangle$ is

$$(7.9) \quad P_n = |\langle n|\alpha\rangle|^2 = \exp(-\bar{n}) \frac{\bar{n}^n}{n!},$$

where $\bar{n} \equiv \langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$. This is the Poissonian probability distribution mentioned in the previous chapter. A laser that operates sufficiently far above its lasing

threshold, such that its amplitude is stabilized by the non-linear optical process of gain saturation, emits approximately coherent light.

(iii) **Thermal light** is in a way the most random form of light and is therefore also called **chaotic light**. It describes the radiation emitted from a black body source at a fixed temperature T . At zero temperature, the resulting optical field in a single mode is just the vacuum state $|0\rangle$. At elevated temperature, the average number of photons in the cavity mode increases. The random nature and lack of phase relations of the radiation forces us to describe the thermal field with a density matrix $\hat{\rho}_{\text{th}}$ instead of a pure state. The density matrix of thermal light, derived from quantum statistical mechanics, is

$$(7.10) \quad \hat{\rho}_{\text{th}} = \frac{\exp[-\hat{\mathcal{H}}/(kT)]}{\text{Trace}[\exp[-\hat{\mathcal{H}}/(kT)]]} = \sum_{n=0}^{\infty} P_n |n\rangle\langle n|, \quad P_n = (1 - \exp[-\hbar\omega/(kT)]) \exp[-n\hbar\omega/(kT)],$$

where $\sum P_n = 1$. This is an exponential distribution with an average photon number known from Bose-Einstein statistics:

$$(7.11) \quad \bar{n}_{\text{th}} \equiv \text{Trace}[\hat{\rho}_{\text{th}} \hat{n}] = \sum_{n=0}^{\infty} n P_n = \frac{1}{\exp(\frac{\hbar\omega}{kT}) - 1}.$$

Figures 7.1 and 7.2 present different graphical representations of coherent and thermal light.

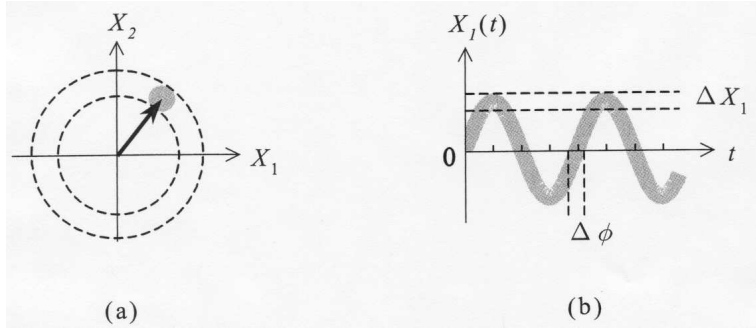


Figure 7.1: A coherent state of light represented in (a) a phasor diagram, and (b) a time trace (Fig. 7.3 of [FOX06]).

7.3 Intensity fluctuations

Although it is somewhat tricky to talk about the intensity fluctuations in a single-optical mode, this topic is often discussed in textbooks. Although the analysis is relatively straightforward in single-mode optics, it still forces us to consider issues

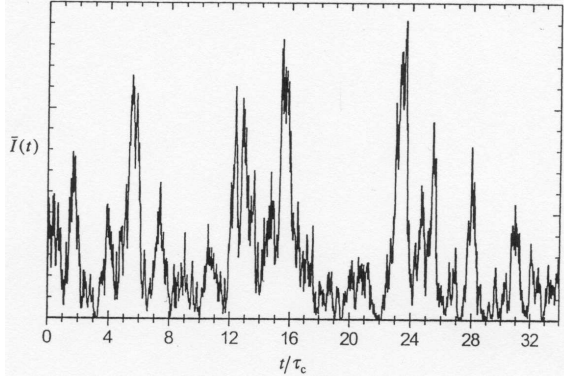


Figure 7.2: The optical intensity emitted by a thermal source (emitting collision-broadened chaotic light) varies wildly on the time scale of the coherence time τ_c (Fig. 3.4 of [LOU03]).

that are also of vital importance for the more complete multi-mode analysis, discussed in the next chapter, which also provides information on the time scale of the fluctuations.

Let's start with the average photon number or average intensity. The quantum theory of photon detection tells us that the average intensity is proportional to $\langle \hat{I}_{\text{det}} \rangle \propto \langle \psi | \hat{E}^- \hat{E}^+ | \psi \rangle \propto \langle \psi | \hat{a}^\dagger \hat{a} | \psi \rangle$. Note that the operators occur only in the so-called **normal ordering**, where any $\hat{a}^\dagger$ operator appears in front of $\hat{a}$. The anti-normal combination $\langle \psi | \hat{a} \hat{a}^\dagger | \psi \rangle$, which produces a non-zero outcome even for the vacuum state, is excluded. It is easy to calculate the average photon number of the three important single-mode quantum states. We find (i) $\langle n | \hat{n} | n \rangle = n$ for the number state $|n\rangle$, (ii) $\langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2$ for the coherent state $|\alpha\rangle$, and (iii) $\langle \hat{n} \rangle = \sum n P_n$ for thermal light.

For the calculation of the intensity fluctuations, the operator ordering is even more important. A quantum mechanical description of the HBT (Hanbury Brown & Twiss) experiment shows that the observed coincidence count rate is again proportional only to the normally-ordered combination of field operators. This has an obvious reason: one can only observe two photon-induced clicks when two or more photons are present. After normalization, we thus obtain

$$(7.12) \quad g^{(2)} = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = \frac{\langle : \hat{n} \hat{n} : \rangle}{\langle \hat{n} \rangle^2} = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2} = 1 + \frac{F - 1}{\langle \hat{n} \rangle},$$

where the combination $::$ denotes normal ordering, and where we have used the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. Note that we have labeled the normalized second-order coherence as $g^{(2)}$ instead of $g^{(2)}(\tau = 0)$ to stress that the single-mode treatment cannot describe the time dependence of the fluctuations. In the last equation, we have defined the operator $\Delta \hat{n}^2 \equiv \hat{n}^2 - \langle \hat{n} \rangle^2$ to link the second-order coherence $g^{(2)}$ to Fano's factor F , introduced in the previous chapter, as

$$(7.13) \quad F \equiv \frac{\langle \Delta \hat{n}^2 \rangle}{\langle \hat{n} \rangle} = 1 + \frac{\langle : \hat{n} \hat{n} : \rangle}{\langle \hat{n} \rangle} - \langle \hat{n} \rangle.$$

For the three quantum states of light discussed in this chapter we calculate the following intensity fluctuations:

(i) For the number state $|n\rangle$, the relation $\langle \hat{n}^2 \rangle = n^2$ yields the intuitive result $\langle \Delta \hat{n}^2 \rangle = 0$, $F = 0$, and $g^{(2)} = 1 - (1/n)$. The later result corresponds to perfect anti-bunching ($g^{(2)} = 0$) only when $n = 1$; number states with $n > 1$ can produce coincidence counts in a HBT experiment!

(ii) For the coherent state $|\alpha\rangle$, the normally-ordered relation $\langle \alpha | : \hat{n}^2 : | \alpha \rangle = |\alpha|^4$ yields $\langle \Delta \hat{n}^2 \rangle = \langle \hat{n} \rangle = |\alpha|^2$, $F = 1$, and the easy-to-remember $g^{(2)} = 1$.

(iii) For thermal light, the relation $\langle n^2 \rangle = \sum n^2 P_n$ in combination with the P_n values presented as Eq. (7.10), yields $\langle \Delta \hat{n}^2 \rangle = \langle \hat{n} \rangle^2 + \langle \hat{n} \rangle$, $F = 1 + \langle \hat{n} \rangle$, and the easy-to-remember $g^{(2)} = 2$. These expectation values can, among others, be calculated by taking the z -derivative of the generating function $G(z) \equiv \sum P_n z^n$ at $z = 1$, where $G(z = 1) = 1$, $G'(z = 1) = \langle n \rangle$, and $G''(z = 1) = \langle n(n - 1) \rangle$.

The three different results mentioned above correspond to (i) photon anti-bunching ($F < 1$), (ii) uncorrelated photons ($F = 1$), and (iii) photon bunching ($F > 1$). The observation of anti-bunching ($F < 1$) always requires a quantum description of the optical field, whereas bunching ($F > 1$) can also be explained in classical terms.

After this short quantum description of intensity fluctuations, we can evaluate the influence of loss on the intensity fluctuations, introduced in Chapter 6, in a more quantitative way. We start by noting that any loss is unavoidably accompanied by quantum noise. The argument is simple: We quantify the loss by its associated amplitude transmission γ (intensity transmission $T = |\gamma|^2$) and write the output field operator as

$$(7.14) \quad \hat{a}_{\text{out}} = \gamma \hat{a}_{\text{in}} + \hat{f}.$$

The commutation relation of the output field is

$$(7.15) \quad [\hat{a}_{\text{out}}, \hat{a}_{\text{out}}^\dagger] = |\gamma|^2 [\hat{a}_{\text{in}}, \hat{a}_{\text{in}}^\dagger] + [\hat{f}, \hat{f}^\dagger],$$

as $[\hat{a}_{\text{in}}, \hat{f}^\dagger] = [\hat{f}, \hat{a}_{\text{in}}^\dagger] = 0$ on account of their uncorrelated nature. As the commutation relation of the quantum field must remain equal to unity, we need $[\hat{f}, \hat{f}^\dagger] = (1 - |\gamma|^2) = 1 - T$. The later equation specifies the “strength of the quantum noise (= vacuum fluctuations) associated with the loss”; it is a simple version of the **quantum mechanical fluctuation-dissipation theorem**.

In order to compare the Fano factor of the input and output fluctuation, we will use the equation that contains only normally-ordered operators, as the expectation value of the vacuum fluctuations is zero under normal ordering. Using Eq. (7.13), we thus quickly find $(F_{\text{out}} - 1) = T(F_{\text{in}} - 1)$. Substitution in Eq. (7.12) yields

the, maybe somewhat surprising but very comforting, result $g_{\text{out}}^{(2)} = g_{\text{in}}^{(2)}$. Hence, the losses do not affect the normalized second-order coherence function $g^{(2)}$.

Allow me to finish this section with a somewhat philosophical remark. For me, the difference between thermal and coherent light remains intriguing. Thermal light, sometimes also called chaotic light, is “maximally random” in terms of its optical field, which has a complex Gaussian distribution function, but is highly structured (= bunched) on the photon level. Coherent light, on the other hand, is highly structured in its optical amplitude, which is more or less constant, but is “maximally random” on the photon level, where its photons behave as “independent particles”. The origin of this apparent controversy seems to lie in the quadratic (=nonlinear) relation between the optical intensity $I \propto |E|^2$ (related to photon number) and the optical field $\hat{E}$, which is described as a quantum operator.

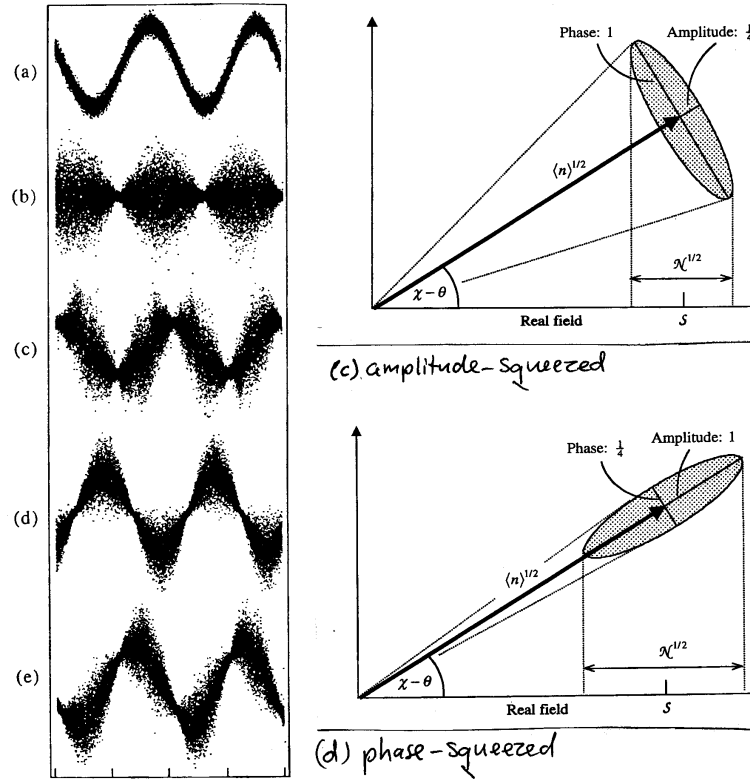


Figure 7.3: Measured electric field of (a) a coherent state, (b) a squeezed vacuum state, (c) an amplitude-squeezed state, (d) a phase-squeezed state, and (e) a squeezed state with 48° between the coherent vector and the axis of the noise ellipse. The scale on the horizontal axis indicates the local oscillator phase (see ref. [BRE97]). Also shown are the phasor plots of the amplitude-squeezed and phase-squeezed state. (Figs. 5.11, 5.13, and 5.14 of [LOU03]).

7.4 Field quadratures and squeezed states

The electro-magnetic field in a single discrete mode behaves as a quantum mechanical oscillator of which the position x and momentum p can be associated with the cosine and sine components of the EM field. The creation and annihilation operators are (scaled) linear combinations of the form $\hat{a} = \hat{x} + i\hat{p}$ and $\hat{a}^\dagger = \hat{x} - i\hat{p}$. These forms naturally link the annihilation operator $\hat{a}$ to the $\exp(-i\omega t)$ component of the optical field and creation operator $\hat{a}^\dagger$ to its $\exp(i\omega t)$ component. In other words, the lowering operator $\hat{a}$ and raising operator $\hat{a}^\dagger$ are associated with the positive and negative frequency part of the electro-magnetic field.

The **quadrature operators** $\hat{X}_1 \equiv \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$ and $\hat{X}_2 \equiv -\frac{1}{2}i(\hat{a} - \hat{a}^\dagger)$ are related to the cosine and sine components of the optical field, or the x and p component of corresponding harmonic oscillator. §7.1-7.6 of the book of Fox describes many properties of these quadrature operators. It also introduces the associated amplitude operator $\hat{n} = \sqrt{\hat{X}_1^2 + \hat{X}_2^2} - \frac{1}{2}$ and “phase operator” $\hat{\phi} = \arctan(\hat{X}_2/\hat{X}_1)$, although the phase operator is ill-defined for small photon number n . The mentioned operators obey the uncertainty relations

$$(7.16) \quad \Delta X_1 \cdot \Delta X_2 \geq 1/4 \quad , \quad \Delta n \cdot \Delta \phi \geq 1/2,$$

where $\Delta X_1 \equiv \langle \hat{X}_1^2 \rangle - \langle \hat{X}_1 \rangle^2$ etc. §7.7-7.10 of the book of Fox describes how the (fluctuations) in the field quadratures can be modified to produce either **quadrature-squeezed light**, where for instance ΔX_1 is reduced at the expense of ΔX_2 , or **amplitude-squeezed light**, where Δn is reduced at the expense of $\Delta \phi$.

Detection of quadrature squeezed light always involves interference with a coherent field that acts as local oscillator to define the reference phase $\phi = 0$. The intensity measured after interference is proportional to

$$(7.17) \quad \langle (a_{\text{LO}}^\dagger e^{i\phi} + \hat{a}^\dagger)(a_{\text{LO}} e^{-i\phi} + \hat{a}) \rangle = \langle a_{\text{LO}}^\dagger a_{\text{LO}} \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle (e^{i\phi} a_{\text{LO}}^\dagger \hat{a} + e^{-i\phi} \hat{a}^\dagger a_{\text{LO}}) \rangle.$$

As the important last interference term contains both $\hat{a}$ and $\hat{a}^\dagger$ operators, the amount of squeezing always deteriorates under the influence of loss, even if these losses are due to the limited quantum efficiency of the detector. This makes squeezing experiments notoriously difficult.

Note that some textbooks include a factor of $\sqrt{2}$ in their definition of the quadrature operators. Using the more symmetric forms $\hat{a} = (1/\sqrt{2})(\hat{x} + i\hat{p})$, $\hat{a}^\dagger = (1/\sqrt{2})(\hat{x} - i\hat{p})$, $\hat{X}_1 = (1/\sqrt{2})(\hat{a} + \hat{a}^\dagger)$, and $\hat{X}_2 = (1/\sqrt{2}i)(\hat{a} - \hat{a}^\dagger)$, they obtain the more natural commutator $[\hat{X}_1, \hat{X}_2] = i$ and the relation $\hat{X}_1^2 + \hat{X}_2^2 = (2\hat{n} + 1)$ reminiscent of the quantum harmonic oscillator.

Chapter 8

Multi-mode quantum optics

The two previous chapters presented both a semi-classical and a quantum description of the (statistical properties of the) optical field. The quantum description was, however, limited to a single discrete optical mode. To properly describe the full dynamics of the field fluctuations we have to move from a single-mode to a continuous multi-mode description of the quantum optical field. We do so by introducing either time or frequency into the operator description, using $\hat{a}(t)$ and $\hat{a}(\omega)$ instead of just $\hat{a}$. This chapter goes beyond “§ 8.5 Quantum theory of Hanbury Brown-Twiss experiments” in the book *Quantum Optics* of M. Fox [FOX06]. It is largely based on “Chapter 6: Multimode and continuous-mode quantum optics” in the book *The Quantum Theory of Light* of R. Loudon [LOU03].

8.1 Continuous-mode quantum optics

We consider the frequency/time dynamics of a single optical polarization and a single transverse mode of the optical field. An experimental realization thereof can for instance be obtained by considering the optical field in a single-mode polarization-preserving optical fiber. Loudon introduces the mentioned frequency dependence in the quantum operator description, by starting from the discrete-mode operators $\hat{a}_k$ and taking the limit of box size $L \rightarrow \infty$, where the mode spacing $\Delta\omega = c\Delta k = 2\pi c/L \rightarrow 0$. In this limit he replaces $\sum_k \rightarrow \int d\omega/\Delta\omega$, $\delta_{k,k'} \rightarrow \Delta\omega\delta(\omega - \omega')$ and

$$(8.1) \quad \hat{a}_k \rightarrow (\Delta\omega)^{1/2}\hat{a}(\omega) \quad , \quad \hat{a}_k^\dagger \rightarrow (\Delta\omega)^{1/2}\hat{a}^\dagger(\omega) \quad ,$$

with associated Hamiltonian $\hat{\mathcal{H}} = \int d\omega \hbar\omega \hat{a}^\dagger(\omega)\hat{a}(\omega)$ and commutation relation

$$(8.2) \quad [\hat{a}(\omega), \hat{a}^\dagger(\omega')] = \delta(\omega - \omega').$$

The transition from time to frequency domain is described by the Fourier relations (8.3)

$$\hat{a}(t) = (2\pi)^{-1/2} \int d\omega \hat{a}(\omega) \exp(-i\omega t) \quad , \quad \hat{a}(\omega) = (2\pi)^{-1/2} \int dt \hat{a}(t) \exp(i\omega t) \quad ,$$

and their Hermitian-conjugates for $\hat{a}^\dagger(t)$ and $\hat{a}^\dagger(\omega)$.

These continuous-mode operators allow one to define new concepts, such as the photon flux $\hat{f}(t) \equiv \hat{a}^\dagger(t)\hat{a}(t)$ (in units of [photons/s]) and the average spectral photon flux per angular bandwidth $f(\omega)$ (in units of [photons/s/s⁻¹] i.e. dimensionless). These concepts are again Fourier related and correspond to the classical fluxes as [LOU03]

$$(8.4) \quad f(t) \equiv \langle \hat{f}(t) \rangle = \int d\omega f(\omega) \quad , \quad \langle \hat{a}^\dagger(\omega) \hat{a}(\omega') \rangle = 2\pi f(\omega) \delta(\omega - \omega').$$

The above set of equations and definitions allow one to solve many problems in quantum optics. The field-correlation function of a stationary field can for instance be expressed as

$$(8.5) \quad G^{(1)}(\tau) \equiv \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle = \int d\omega f(\omega) \exp(-i\omega\tau).$$

This important relation, which states that the field-correlation function $G^{(1)}(\tau)$ is simply Fourier-related to the optical spectrum $f(\omega)$, proves that interferometric experiments contain *exactly the same information* as a spectral analysis *for any quantum state of light*. Only the required temporal/frequency resolution and the available equipment will determine the preferred experiment.

8.2 Field and intensity correlations

The **field correlation function** $G^{(1)}(\tau)$ introduced in the previous section is often written in its normalized form

$$(8.6) \quad g^{(1)}(\tau) \equiv \frac{\langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle}{\langle \hat{a}^\dagger(t) \hat{a}(t) \rangle}.$$

This function peaks at $g^{(1)}(0) = 1$ and has a temporal coherence width that is approximately the inverse of the spectral bandwidth of the light.

The quantum version of the **intensity correlation function** is defined as $G^{(2)}(\tau) \equiv \langle : \hat{I}(t) \hat{I}(t + \tau) : \rangle$, and its normalized form

$$(8.7) \quad g^{(2)}(\tau) \equiv \frac{\langle : \hat{I}(t) \hat{I}(t + \tau) : \rangle}{\langle \hat{I}(t) \rangle^2} = \frac{\langle \hat{a}^\dagger(t) \hat{a}^\dagger(t + \tau) \hat{a}(t + \tau) \hat{a}(t) \rangle}{\langle \hat{a}^\dagger(t) \hat{a}(t) \rangle^2},$$

as $\hat{I}(t) \propto \hat{f}(t) = \hat{a}^\dagger(t)\hat{a}(t)$. The normal ordering of the field operators, denoted by the sandwich $::$, is needed to properly describe the photo-detection process.[LOU03] A convenient property of the normally-ordered operators is that their expectation value is insensitive to the vacuum fluctuation that “leak-in” under the influence of losses, i.e., $\langle : \hat{N}^2 : \rangle = T^2 \langle : \hat{n}^2 : \rangle$ under the transformation $\hat{n} \rightarrow \hat{N} = T\hat{n}$. As a result, $g^{(2)}(0)$ is unaffected by losses *for any quantum state of light*.

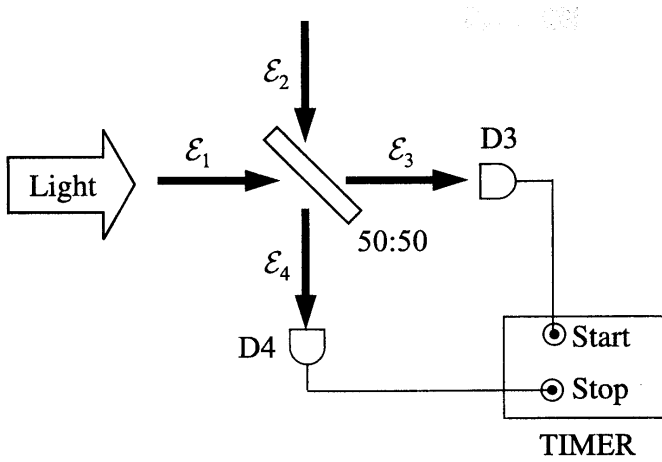


Figure 8.1: Hanbury Brown and Twiss setup with a start-stop time correlator. The incident light $\mathcal{E}_\infty$ is split at a 50:50 beam splitter and detected by single-photon detectors D3 and D4. The count pulses from D3 start an electronic timer that is stopped by a count pulse from D4. Statistical analysis of a series of these events yield the probability distribution of consecutive photo detection events (see text) and the joint probability distribution for detection of two photons separated by a time difference τ .

The intensity correlation function can be measured in a so-called **start-stop correlation experiment**, which records the time intervals between consecutive photo detection event (see Fig. 8.2). The same data can also be used to calculate the joint probability $P(t+\tau; t)$ to detect *any* photon at time $t+\tau$ after the detection of a first photon at time t . The intensity correlation function can also be written as

$$(8.8) \quad g^{(2)}(\tau) = \frac{P(t+\tau; t)}{\lim_{\tau \rightarrow \infty} P(t+\tau; t)}$$

The intensity correlation are very different for the three important quantum fields: (i) thermal light, (ii) coherent light, and (iii) light from a single photon source. These sources exhibit: (i) bunching at $g^{(2)}(0) > 1$ (chaotic light), (ii) Poisson statistics at $g^{(2)}(0) = 1$, and (iii) anti-bunching at $g^{(2)}(0) < 1$.

For thermal light, the (complex) Gaussian statistics of the amplitude fluctuations results in the important moment factorization theorem

$$(8.9) \quad \langle \hat{a}^\dagger(\omega_1)\hat{a}^\dagger(\omega_2)\hat{a}(\omega_3)\hat{a}(\omega_4) \rangle = \langle \hat{a}^\dagger(\omega_1)\hat{a}(\omega_3) \rangle \langle \hat{a}^\dagger(\omega_2)\hat{a}(\omega_4) \rangle + \langle \hat{a}^\dagger(\omega_1)\hat{a}(\omega_4) \rangle \langle \hat{a}^\dagger(\omega_2)\hat{a}(\omega_3) \rangle$$

This theorem allows one to rewrite the normalized second-order correlation function of thermal light in terms of its first-order correlation as

$$(8.10) \quad g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2,$$

thus providing a direct link between the intensity and field fluctuations of thermal light. At $\tau = 0$, the peak value $g^{(2)}(0) = 2$, equivalent to $\langle : \hat{I}(t)^2 : \rangle = 2\langle \hat{I} \rangle^2$, is associated with an exponential probability distribution of the intensity variations.

8.3 The quantum beam splitter

Beam splitters are important in many optical experiments, as they allow one to split and recombine optical beams. The quantum-mechanical description of a beam splitter is more intriguing as one might think as it requires one to include something like the leakage of “quantum noise” through the unused input port. The argument is as follows: Consider a lossless beam splitter with amplitude transmission t and t' and amplitude reflections r and r' for the four different routes from input to output such that

$$(8.11) \quad \begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}.$$

The lossless character of this linear transformation imposes the following unitary relations: $|t|^2 + |r|^2 = |t'|^2 + |r'|^2 = 1$ and $t.r^* + r'.t'^* = 0$. Convenient choices for a 50/50 beam splitter are $t = t' = 1/\sqrt{2}$ in combination with either (i) the symmetric choice $r = r' = i/\sqrt{2}$, or (ii) Fox’s choice $r = 1/\sqrt{2}$ and $r' = -1/\sqrt{2}$, where the difference corresponds to a different choice of reference plane. Whichever choice one makes, one has to include the field operator of the unused port in the quantum description to ensure that the commutation relations $[\hat{a}_i, \hat{a}_i^\dagger] = 1$ are satisfied for all optical ports. The limited transmission through the beam splitter shouldn’t result in a reduction of commutation relations in the output port!

A convenient way to describe the importance of the “empty” beam-splitter port is the statement that the empty port allows vacuum fluctuations to leak into the output beams. In semi-classical terms, the effective strength of these vacuum fluctuations corresponds to *one photon per second per Hz spectral bandwidth*. In quantum-mechanical terms, the commutation relation of the field operator of the open port is given by $[\hat{a}(\omega), \hat{a}^\dagger(\omega')] = \delta(\omega - \omega')$. Vacuum fluctuations, also denoted as quantum noise, are the reason why squeezed light loses part of its squeezing under the influence of loss. Vacuum fluctuations also result in a change in photon statistics under the influence of loss (see chapter 6). Vacuum fluctuations do not affect the normalized intensity correlation function $g^{(2)}(0)$.

Chapter 9

Light-atom interaction 1

The dipolar interaction between light and matter is described by the interaction Hamiltonian $H_{\text{int}} = -\boldsymbol{\mu} \cdot \mathbf{E}$, where $\boldsymbol{\mu}$ is the atomic dipole and $\mathbf{E}$ is the optical field. Despite its simple form, this interaction contains many aspects: the vector character of $\boldsymbol{\mu}$ and $\mathbf{E}$, the density of the available optical modes, the coherence and saturation of the material excitation, and the various damping mechanism of the transition. I hope you recognize some of the concepts that have also been discussed in the course Quantum Mechanics 2.

This chapter presents two simple descriptions of light-atom interaction and a general discussion on the optical frequency response. The material is based on §4.1-4.5 of *Quantum Optics* [FOX06], the book *Laser Electronics* of Verdeyen [VER89], and Chapter 25 of the book *Introduction to Optics (3rd edition)* of Pedrotti et al. [PED07]. The next chapter introduces the Bloch vector description of the atomic transition and extends the discussion to stronger interaction with saturation.

9.1 Density of states (DOS)

Appendix C of ref. [FOX06] describes how to calculate the spectral density of the optical modes, i.e., the number of modes per unit volume per unit spectral range in units [$\text{m}^{-3}/\text{s}^{-1}$]). The described counting procedure starts by placing a fictitious box around a volume V and imposing periodic boundary conditions on the modes, such that the wave vector of each mode is $\mathbf{k} = (N_x \frac{2\pi}{L_x}, N_y \frac{2\pi}{L_y}, N_z \frac{2\pi}{L_z})$ (see also chapter 7 of this syllabus and Fig. 9.1 for a simplified 2D version). The number of modes per interval dk in $\mathbf{k}$ -space is easily found to be $2 \times 4\pi k^2 dk / [(2\pi)^3/V] = V k^2 / \pi^2$, where the factor two originates from the two polarizations, where the factor $4\pi k^2$ is the surface of a spherical shell, and where $(2\pi)^3/V$ is the k -space volume per mode.

This result easily can be expressed as a spectral density of states (DOS) by using the general relations $k = n\omega/c$ and $dk = n_g d\omega/c$, where n is the refractive index and $n_g(\omega) = n + \omega[\partial n/\partial \omega]$ is the group refractive index at frequency ω . We thus find an **optical mode density** or **density of states (DOS)** [VER89]

$$(9.1) \quad p(\omega) = n^2 n_g \frac{\omega^2}{\pi^2 c^3},$$

in units $[m^{-3}/s^{-1}]$. We generally limit ourselves to the case $n = n_g = 1$. Outside this limit, the extra factor n^2 accounts for the *transverse or angular compression* of the radiation upon entering a medium with a higher refractive index (Snell's law of refraction). The extra factor n_g accounts for the *longitudinal compression* of the energy density associated with the (generally reduced) group velocity $v_g = c/n_g$. Inside a medium, the energy density does not only reside in the electro-magnetic field in between the atoms, but is also stored in the atomic excitation, c.q. the polarization.

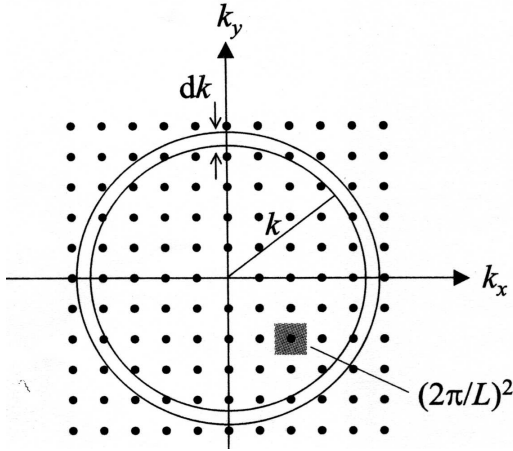


Figure 9.1: The density of optical modes $p(\omega)$ inside a box can be calculated with a simple counting procedure. This figure depicts a two-dimensional version of this mode counting with mode spacings $(2\pi/L)$ in each direction in k -space. (Fig. C2 from Appendix C of ref. [FOX06])

The density of states, described by (9.1), is an essential ingredient to understand **Planck's law** for the spectral energy density of a black-body source

$$(9.2) \quad u(\omega) = \hbar\omega p(\omega) n_{th} = \hbar\omega \left(\frac{\omega^2}{\pi^2 c^3} \right) \frac{1}{\exp[\hbar\omega/(k_B T)] - 1},$$

where $k_B = 1.3810^{-23} J/K$ is Boltzmann's constant. This expression contains three factors: $\hbar\omega$ is the energy per photon, $\omega^2/(\pi^2 c^3)$ is the spectral density of optical modes, and $n_{th} = 1/(\exp[\hbar\omega/(k_B T)] - 1)$ is the average photon number per mode at temperature T , as given by the **Bose-Einstein statistics of the photons**.

9.2 Einstein's A and B coefficients

One of the simplest descriptions of the light-atom interaction was given by Einstein. He described the interaction between an ensemble of two-level atoms in equilibrium with a thermal optical field with simple **rate equations**, thus neglecting the material coherence (c.q. polarization). Einstein's rate equations are

$$(9.3) \quad \frac{dN_2}{dt} = -A_{21}N_2 - B_{21}^\omega u(\omega)N_2 + B_{12}^\omega u(\omega)N_1 = -\frac{dN_1}{dt},$$

for the number of atoms N_2 and N_1 in the upper and lower level, respectively. The optical transition is assumed to be closed, making $N_1 + N_2$ constant. Three optical processes contribute to the population transfer: The Einstein A-coefficient $A_{21} = 1/\tau_{\text{rad}}$ is the spontaneous emission rate, where τ_{rad} is the radiative lifetime of the upper level. The two Einstein B-coefficients quantify the strength of the optical absorption (B_{12}^ω) and stimulated emission (B_{21}^ω). The associated transition rates (in units $[s^{-1}]$) are found by multiplication the B-coefficients by the spectral energy density $u(\omega)$ (in units of $[J.m^{-3}/s^{-1}$ radial spectral bandwidth]).

Einstein used thermodynamic arguments to find relations between his A and B coefficients. He basically solved Eq. (9.3) and compared the steady-state result

$$(9.4) \quad u(\omega) = \frac{A_{21}}{(N_1/N_2)B_{12}^\omega - B_{21}^\omega}$$

with the known Boltzman distribution over the atomic levels $N_2/N_1 = \exp -\hbar\omega/(kT)$ and Planck's distribution over the photon occupancy (see Eq.(9.2)). For a simple two-level system he thus found $B_{12}^\omega = B_{21}^\omega$ and

$$(9.5) \quad A_{21} = \left(\frac{\hbar\omega^3}{\pi^2 c^3} \right) B_{21}^\omega.$$

For a slightly more complicated system, comprising g_1 frequency-degenerate ground states and g_2 frequency-degenerate excited states, the high-power balance changes into $g_1 B_{12}^\omega = g_2 B_{21}^\omega$, while Eq. (9.5) remains identical.

With the above equations, Einstein not only introduced the concept of stimulated emission, as a logical counterpart of absorption, but also linked spontaneous and stimulated emission. In somewhat sloppy language one might say that "spontaneous emission is like stimulated emission that is stimulated by vacuum fluctuations". By comparing Eqs. (9.2)-(9.5) one finds that the "strength of the vacuum fluctuations" correspond to "one photon per optical mode". This tentative formulation originates from the commutation relation of the field operators $\hat{a}$ and $\hat{a}^\dagger$.

9.3 Radiative transition rates: quantum treatment

Section 4.2 of ref. [FOX06] gives a microscopic description of the spontaneous emission rate $A_{21} = 1/\tau_{\text{rad}}$ from a quantum-mechanical point of view. It calculates the radiative decay rate from the upper to the lower level from Fermi's golden rule,

$$(9.6) \quad A_{21} = \frac{2\pi}{\hbar} |M_{12}|^2 g(\hbar\omega).$$

The derivation comprises three crucial steps. First, we note that Fermi's golden contains the density of states per energy unit, which relates to the density of states per unit volume and angular bandwidth as $g(\hbar\omega) = p(\omega) \cdot V/\hbar$. Second, the dipole interaction Hamiltonian $H_{\text{int}} = -\boldsymbol{\mu}_{12} \cdot \mathbf{E}$ contains the inner product of two vectors, being the electric dipole moment $\boldsymbol{\mu}_{12}$ and the electric field $\mathbf{E}$. Hence, the interaction strength depends on the relative orientation between these vectors. It is common to average over all possible orientations and write $\langle |\boldsymbol{\mu}_{12} \cdot \mathbf{E}|^2 \rangle = (1/3) |\boldsymbol{\mu}_{12}|^2 \langle |\mathbf{E}|^2 \rangle$, using $\langle \cos^2 \theta \rangle = 1/3$. As a third and final step, we use the idea that “spontaneous emission is like stimulated emission that is stimulated by vacuum fluctuations” and quantify the strength of the vacuum fluctuations as $\langle |\mathbf{E}|^2 \rangle = \hbar\omega/(2\epsilon_0 V)$ per mode (see Chapter 7 of this syllabus). By combining these three steps one arrives at the expression:

$$(9.7) \quad A_{21} = \frac{\omega^3}{3\pi\epsilon_0 \hbar c^3} |\boldsymbol{\mu}_{12}|^2.$$

9.4 The classical Lorentz dipole model

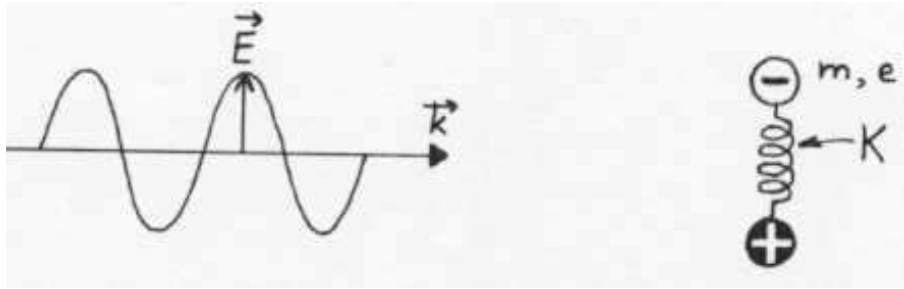


Figure 9.2: *Lorentz model of oscillating dipole excited by an incident optical field.*

Lorentz described the oscillation of a bound electron under the influence of an oscillating electric field with a simple classical model. He modelled the displacement $x(t)$ of an electron bound to a nucleus as a damped harmonic oscillator

$$(9.8) \quad \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{-eE}{m},$$

where E is the driving electric field, m and $-e$ are the electron mass and charge, γ is the damping rate, and $\omega_0^2 = K/m$ is the natural resonance of the bound electron under the influence of a restoring force $F = -Kx$. Under excitation with a monochromatic field of the form $E(t) = \text{Re}[E_0 \exp(-i\omega t)]$, the electron oscillates with the same frequency but a potentially different phase as $x(t) = \text{Re}[x_0 \exp(-i\omega t)]$. A simple Fourier transformation of Eq. (9.8) yield the complex displacement amplitude x_0 and the associated dipole moment of the driven oscillation

$$(9.9) \quad \mu_0 = -ex_0 = \frac{e^2 E_0 / m}{\omega_0^2 - \omega^2 - i\omega\gamma} \approx \left(\frac{-e^2 E_0}{m\omega_0\gamma} \right) \frac{1}{\Delta + i},$$

where the final expression assumes relatively weak damping ($\gamma \ll \omega_0$) and where $\Delta \equiv 2(\omega - \omega_0)/\gamma$ is the normalized detuning. These expressions have a complex Lorentzian resonance structure with a width (FWHM) of $\Delta\omega = \gamma$. They model the optical absorption spectrum of a single bound electron. At resonance ($\Delta = 0$), the susceptibility χ is purely imaginary. At large negative detuning ($\omega - \omega_0 \ll \gamma$), the dominantly real-valued $\chi > 0$ corresponds to in-phase oscillation. At large positive detuning ($\omega - \omega_0 \gg \gamma$), $\chi < 0$ corresponds to out-of-phase oscillations.

The Lorentz dipole model also allows one to estimate the radiative damping rate of the excitation. For this, we consider the natural evolution of the amplitude after excitation $x(t) = x_0 \exp[-(\gamma/2)t - i\omega't]$. The radiative energy loss rate of a classical oscillating dipole $\mu(t) = \text{Re}[\mu_0 \exp(-i\omega t)]$, as calculated from Maxwell's equations, is [JAC75]

$$(9.10) \quad P_{\text{rad}} = \frac{1}{4\pi\epsilon_0} \frac{\omega_0^4 |\mu_0|^2}{3c^3}.$$

By combining this equation with the expression for the combined potential and kinetic energy of the oscillating charge, $U = \langle \frac{1}{2}m(dx/dt)^2 + \frac{1}{2}Kx^2 \rangle = \frac{1}{2}(m\omega_0^2 |\mu_0|^2 / e^2)$, one immediately obtains the classical estimate of the spontaneous emission rate

$$(9.11) \quad A_{\text{classical}} = \gamma_{\text{rad}} \equiv \frac{1}{\tau_{\text{rad}}} = \frac{P_{\text{rad}}}{U} = \frac{e^2 \omega_0^2}{6\pi\epsilon_0 mc^3}.$$

The radiative lifetime of this idealized system is $\tau_{\text{rad}} \approx 45\lambda_0^2$ if we express τ_{rad} in ns and λ_0 in μm , yielding radiative lifetimes of 8-30 ns for transitions at optical frequencies.

Next we compare the quantum-mechanic expression for the spontaneous decay rate of Eq. (9.7) with the the just-derived classical result of Eq. (9.11). This comparison provides for a definition of the so-called **oscillator strength** of the transition

$$(9.12) \quad f_{21} = \frac{1}{3} \frac{A_{21,\text{QM}}}{A_{21,\text{classical}}} = \frac{1}{3} \left(\frac{\omega_0^3}{3\pi\epsilon_0 \hbar c^3} |\boldsymbol{\mu}_{12}|^2 \right) / \left(\frac{e^2 \omega_0^2}{6\pi\epsilon_0 mc^3} \right) = \left(\frac{2m\omega_0}{3e^2 \hbar} \right) |\boldsymbol{\mu}_{12}|^2.$$

The factor $1/3$ is almost a matter of definition; it is related to the m-degeneracy of the quantum levels and is chosen such that the three $p \rightarrow s$ transitions from the $m \pm 1, 0$ ($l = 1$) upper levels to the $m = l = 0$ ground state each have oscillator strength $f = 1/3$. Only strongly-allowed optical transitions, with dominant radiative decay, have $\sum f \approx 1$ after summing over all m -levels. Weak transitions have $f \ll 1$. As a curiosity, we note that the oscillator strengths of all transition starting from the ground state of an atom obey the sum rule $\sum_j f_{ji} = M$ if the atom has M valence electrons and if all transitions are dominated by radiative decay.

Let's return to the driven system and model the atoms in any medium as a set of Lorentz dipoles with an effective density $\mathcal{N} \equiv N/V$, where $\mathcal{N} = \mathcal{N}_1 - \mathcal{N}_2$ is the difference between the ground-state and excited-state population densities. The relative dielectric constant ϵ_r of this medium can be calculated by writing its optical polarization as $P = N\mu$ and using

$$(9.13) \quad \epsilon_r \equiv (n + i\kappa)^2 \equiv 1 + \chi \equiv 1 + \frac{P}{\epsilon_0 E} = 1 + \left(\frac{-\mathcal{N}e^2}{m\omega_0\gamma\epsilon_0} \right) \frac{1}{\Delta + i},$$

where n and κ are the real and imaginary part of the complex refractive index, where χ is the electric susceptibility, and where P and E are complex. The optical response of media with more than one optical resonance can be modeled by a simple summation over complex Lorentzian resonances as

$$(9.14) \quad (n + i\kappa)^2 = 1 + \sum_i \frac{-A_i}{i + (\omega - \omega_i)T_{2,i}},$$

where A_i , ω_i , and $T_{2,i}$ are the resonance strengths, frequencies, and damping times, respectively.

Equation (9.13) can be rewritten in an alternative and very convenient form. By substitution of the classical expression for the radiative lifetime, Eq. (9.7), we easily find

$$(9.15) \quad \chi = \frac{3N\lambda_0^3}{4\pi^2} \left(\frac{\gamma_{\text{rad}}}{\gamma} \right) \frac{-1}{i + \Delta},$$

where γ_{rad} is the radiative decay rate and $\gamma = \gamma_{\text{rad}} + \gamma_{\text{NR}} + 2\gamma_{\text{pure}}$ is the total decay rate of $|P|^2$. This total decay includes the **non-radiative energy decay** γ_{NR} as well as the **pure dephasing** γ_{pure} , associated with random phase changes of the oscillation as induced for instance by collisions in the gas phase or environmental changes in the liquid or solid state. When the above expression is rewritten in terms of the resonant **absorption cross section** σ per atom/molecule it yields the intriguing result

$$(9.16) \quad \sigma_{\text{max}} = 3 \frac{\lambda_0^2}{2\pi} \frac{\gamma_{\text{rad}}}{\gamma},$$

in a medium with effective index $n = 1$. The factor 3 is specific for the absorption from a single $m = 0$ ground state to three $m = \pm 1, 0$ excited states; a special case for which the total absorption (summed over the 3 m-levels) does not depend on the polarization of the optical field. We thus find that the optical absorption cross section is determined by the optical wavelength rather than the size of the atom/molecule! The definition of the optical cross section σ is such that the inverse intensity absorption length in a medium with a ground-state density $\mathcal{N}$ is $\alpha = \mathcal{N}\sigma$. The absorption/gain in a medium with refractive index n can be calculated by replacing λ_0 by $\lambda \equiv \lambda_0/n$ and by replacing the ground-state density $\mathcal{N}$ by the population difference $\Delta\mathcal{N} \equiv \mathcal{N}_1 - \mathcal{N}_2$.

9.5 Transition selection rules

The vector nature of the electric dipole moment

$$(9.17) \quad \mu_{12} = -e\langle 1|\mathbf{r}|2\rangle = -e \int \mathbf{r}\phi_1^*(\mathbf{r})\phi_2(\mathbf{r})dx dy dz$$

imposes important symmetry restrictions on the optical interaction. These so-called **selection rules** are different for single-electron atoms, where the single-electron state is labeled by the quantum numbers l, m, s and m_s , than for multi-electron atoms, where the multi-electron states are labeled by the quantum numbers L, S, J and M_J . Please read Section 4.3 of the book of Fox [FOX06] for an extensive discussion on the selection rules.

To understand the mentioned selection rules, we briefly recall the labeling of the atomic levels. Single-electron states are labeled by their spin s , their orbital angular momentum l , their combined angular momentum $j = |\mathbf{j}| = |\mathbf{l} + \mathbf{s}|$ and its projection m_j on a chosen quantization axis. In a strong magnetic field, the field-induced energy splitting can overrule the spin-orbit interaction and rearrange levels that were originally labeled by (j, m_j) into new levels that are now labeled by the quantum numbers m_l and m_s , being the orbital and spin angular momentum projected on the axis defined by the magnetic field.

Multi-electron states with strong $L - S$ coupling are labeled by their combined spin $S = |\mathbf{S}| = |\sum \mathbf{s}_i|$, their combined orbital angular momentum $L = |\mathbf{L}| = |\sum \mathbf{l}_i|$, their total angular momentum $J = |\mathbf{J}| = |\mathbf{L} + \mathbf{S}|$ and its projection M_J on the quantization axis. These levels are typically labeled as $N^{(2S+1)}L_J$, where the N indicates the electronic shell, where the superscript $(2S+1)$ denotes the multiplicity of the levels, where the central letter denotes the orbital angular momentum L (indicated by $S, P, D, ..$ for $L = 0, 1, 2, ..$), and where the subscript denotes the J . The combination $(2S+1)$ is called the multiplicity of the spin states, where $S = 0$

is a singlet, $S = 1/2$ is a doublet, $S = 1$ is a triplet, etc. With this labeling the ground state of atomic Hydrogen is a (doublet) $1^2S_{1/2}$, while its excited states are a doublet $2^2S_{1/2}$ (for the excitation to the 2s shell), and $2^2P_{1/2}$, and $2^2P_{3/2}$ (for the excitation to the 2p shell). The selection rules state that the optical transition $2^2S_{1/2} \rightarrow 1^2S_{1/2}$ is forbidden, but the transitions $2^2P_{1/2,3/2} \rightarrow 1^2S_{1/2}$ are allowed. Likewise, the ground state of atomic Helium is a (singlet) 1^1S_0 , while its excited states are 2^1S_0 and 2^3S_1 (for the excitation of one electron to the 2s shell) and 2^1P_1 and $2^3P_{0,1,2}$ (for the excitation of one electron to the 2p shell). The selection rules state that the s-singlet transition $2^1S_0 \rightarrow 1^1S_0$ is forbidden and that the s-triplet-to-singlet transition $2^3S_0 \rightarrow 1^1S_0$ is even doubly forbidden. The only allowed transitions are the p-singlet transition $2^1P_1 \rightarrow 1^1S_0$ and the p-triplet transitions $2^3P_{0,1,2} \rightarrow 1^3S_1$. All these transitions should of course also satisfy the selection rule for the projected momentum, which reads $\Delta M_J = \pm 1$ for optical excitations along the quantization axis and $\Delta M_J = 0, \pm 1$ for optical propagation in different directions. The electric field component parallel to the quantization axis interacts solely with the $\Delta M_J = 0$ transition, while the electric field components orthogonal to the quantization axis decompose in circularly-polarized fields that address the $\Delta M_J = \pm 1$ transitions.

Finally we note that electric dipole transitions are not the only optical transitions that are possible. A simple derivation shows that there are other (dipole-forbidden) transitions that can also occur, albeit with much lower probabilities. This derivation is based on the idea that dipole-forbidden transitions would only be strictly forbidden if the EM field would be uniform over the atom. In general, the transition element associated with the atom-field interaction can be expanded as

$$(9.18) \quad \langle f | H_{\text{int}} | i \rangle = \langle f | -e\mathbf{r}\mathbf{E}(\mathbf{r}) | i \rangle \approx \langle f | -e\mathbf{r}\mathbf{E}(\mathbf{0}) | i \rangle + \langle f | -e\mathbf{r}(\mathbf{i}\mathbf{r} \cdot \mathbf{k})\mathbf{E} | i \rangle,$$

where $\mathbf{k}$ is the wavevector of the EM field. The dominant term in this expansion is associated with the strong **electric-dipole transition**. The second term in this expansion is associated with a combination of an **electric-quadrupole and a magnetic-dipole transition**. [JAC75] The relative amplitude of these terms, as compare to the dominant term, is of the order of kr , being the size of the atom as compared to the optical wavelength divided by 2π . From a more fundamental point of view the relative weight of these terms is given by the fine structure constant

$$(9.19) \quad \alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

The transition rates of dipole-forbidden transitions, which disobey the usual selection rules, is typically a factor $\alpha^2 \approx 10^{-4}$ smaller than that of the allowed transitions.

Chapter 10

Light-atom interaction 2

This chapter extends the previous discussion of light-atom interaction by keeping track of the optical coherence and including optical saturation. It introduces the Bloch vector description as a natural tool to describe the evolution of the atomic state under the influence of various atomic decay processes. This material is based on chapter 9 of *Quantum Optics* [FOX06].

10.1 Quantum description of atom-field interaction

We describe the evolution of the quantum state of a single two-level atom as

$$(10.1) \quad |\psi(t)\rangle = c_1(t)e^{-iE_1t/\hbar}|1\rangle + c_2(t)e^{-iE_2t/\hbar}|2\rangle ,$$

where we already singled out the slowly-varying (complex) probability amplitudes $c_i(t)$ from the fast oscillation at transition frequency $\omega_0 \equiv (E_2 - E_1)/\hbar$, and where E_i are the energies of the two levels $i = \{1, 2\}$. Sections 9.3- 9.5 of ref. [FOX06] describe the evolution of the state amplitudes $c_i(t)$ under optical excitation. The energy shift of the atomic dipole in the electric field, also denoted as the **AC-Stark shift**, is described by the interaction potential

$$(10.2) \quad V(t) = -\mu_{12}E(t) = -\mu_{12}E_0 \cos \omega t = -\frac{1}{2}\mu_{12}E_0 (e^{-i\omega t} + e^{i\omega t}) .$$

Substitution of Eqs. (10.1) and (10.2) in the Schrödinger equation $i\hbar d|\psi(t)\rangle/dt = [H_0 + V(t)]|\psi(t)\rangle$ yields

$$(10.3) \quad \frac{d}{dt}c_1(t) = \frac{i}{2}\Omega_R e^{i(\omega-\omega_0)t}c_2(t)$$

$$(10.4) \quad \frac{d}{dt}c_2(t) = \frac{i}{2}\Omega_R e^{-i(\omega-\omega_0)t}c_1(t) .$$

To obtain this simple result, we used the so-called **rotating-wave approximation**, which neglects terms that oscillates at frequencies $\approx 2\omega_0$, and introduced the Rabi frequency

$$(10.5) \quad \Omega_R \equiv |\mu_{12}E_0/\hbar|.$$

The state evolution under excitation with a resonant optical field at $\omega = \omega_0$ is a simple periodic oscillation of the form $c_1(t) = \cos \frac{1}{2}\Omega_R t$ and $c_2(t) = i \sin \frac{1}{2}\Omega_R t$. This evolution corresponds to a period exchange of the ground-state population $|c_1(t)|^2$ and excited-state population $|c_2(t)|^2$ at a frequency Ω_R ; it is called a **Rabi oscillation** or **Rabi flopping**. When the excitation frequency is off-resonant with the optical transition, the excited-state population will oscillate less deep and at a faster rate $\Omega \equiv \sqrt{\Omega_R^2 + \delta\omega^2}$, where $\delta\omega = \omega - \omega_0$ is the frequency detuning.

10.2 Weak excitation and optical absorption

Under weak excitation, where the ground-state population remains at $|c_1|^2 \approx 1$, a simple integration of Eq. (10.4) directly yields the rate of optical absorption and the associated Einstein B_{12} coefficient. The derivation of this relation is as follows (see §9.4 of [FOX06]). The mentioned integration yields the excited-state population

$$(10.6) \quad |c_2(t)|^2 = \left(\frac{\Omega_R}{2}\right)^2 \left(\frac{\sin[\frac{1}{2}(\omega - \omega_0)t]}{\frac{1}{2}(\omega - \omega_0)}\right)^2,$$

for excitation with a mono-chromatic optical field at frequency ω . At resonance we obtain the seemingly surprising result that the excited-state population increased *quadratically in time*, being the start of the cos-type time dependence typical for Rabi oscillations. This result changes for excitation with an optical field with a sufficiently broad spectral width. Integration of Eq. (10.6) over a broad spectrum now yields a *linear time dependence* of the form

$$(10.7) \quad |c_2(t)|^2 = \frac{\pi}{\epsilon_0 \hbar^2} |\mu_{12}|^2 u(\omega_0) t,$$

where $u(\omega_0)$ is the spectral energy density introduced in the previous chapter. A similar trick with spectral integration was used in the derivation of Fermi's golden rule. Equation (10.7) presents a microscopic model for Einstein's B_{12} coefficient for optical absorption, which yields the correct form

$$(10.8) \quad B_{12}^\omega = \frac{\pi |\boldsymbol{\mu}_{12}|^2}{3\epsilon_0 \hbar^2},$$

after inclusion of a factor $\langle |\cos \theta|^2 \rangle = 1/3$ to account for randomly oriented dipoles.

10.3 State evolution and damping

To properly describe the coherence of an atom, or an ensemble of atoms, it is often more convenient to work with the atomic density matrix $\rho(t)$ instead of the quantum state $|\psi(t)\rangle$. The elements of this 2×2 density matrix are defined as $\rho_{ij}(t) \equiv \langle c_i(t)c_j^*(t) \rangle$, where the symbol $\langle \rangle$ denotes ensemble averaging. The on-diagonal elements ρ_{ii} correspond to the atomic populations; the off-diagonal elements of ρ_{ij} correspond to the atomic coherence.

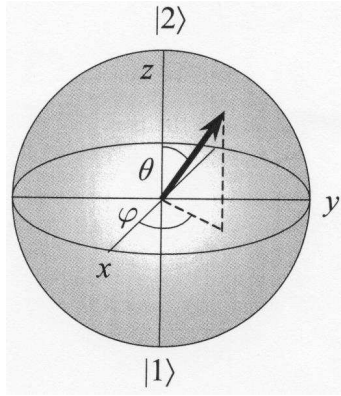


Figure 10.1: The Bloch vector $\mathbf{S}$ presents a convenient representation of the population and atomic coherence of a two-level system. The vertical component S_z denotes the population difference, the horizontal component $S_x + iS_y$ denotes the atomic coherence. (Fig. 9.9 of ref. [FOX06])

The atomic coherence and population can be conveniently combined in a so-called **Bloch vector** $\mathbf{S} = (S_x, S_y, S_z)$ with coefficients

$$(10.9) \quad S_x + iS_y = 2\langle c_1 c_2^* \rangle, \quad S_z = \langle |c_2|^2 \rangle - \langle |c_1|^2 \rangle.$$

The Bloch vector combines the non-trivial coefficients in the expansion of the density matrix in terms of the Pauli matrices σ_i , as

$$(10.10) \quad \rho = \frac{1}{2}\sigma_0 + \sum_{i=\{x,y,z\}} \frac{1}{2}S_i\sigma_i$$

$$(10.11) \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Figure 10.1 shows the construction of the Bloch vector. Pure quantum states correspond to Bloch vectors on the unit sphere ($|\mathbf{S}| = 1$).

The density matrix description allows one to easily distinguish between two different forms of atomic damping. The population difference $\rho_{22} - \rho_{11}$, associated with the on-diagonal elements of ρ , decays via **longitudinal relaxation** (or **energy relaxation**) at a rate $\gamma_{\parallel} = 1/T_1$, where T_1 is the population decay rate. The atomic coherence, associated with the off-diagonal matrix elements $\rho_{12} = \rho_{21}^*$, decays via

transverse relaxation at a rate $\gamma_{\perp} = 1/T_2$, where T_2 is the dephasing rate. The relation between these rates is

$$(10.12) \quad \gamma_{\perp} = \frac{1}{2}\gamma_{\parallel} + \gamma' \quad , \quad \frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_2'} ,$$

where the factor $\frac{1}{2}$ stems from the difference between amplitude and intensity decay. The decay rate $\gamma' = 1/T_2'$ accounts for **pure dephasing** by population-conserving interaction. Potential mechanisms are population-conserving collisions in the gas phase and environmental changes in the liquid or solid state. The transverse component ($S_x + iS_y$) of the Bloch vector decays at a rate $\gamma_{\perp}$, while its longitudinal component S_z decays at a rate $\gamma_{\parallel}$ (see Fig. 10.2).

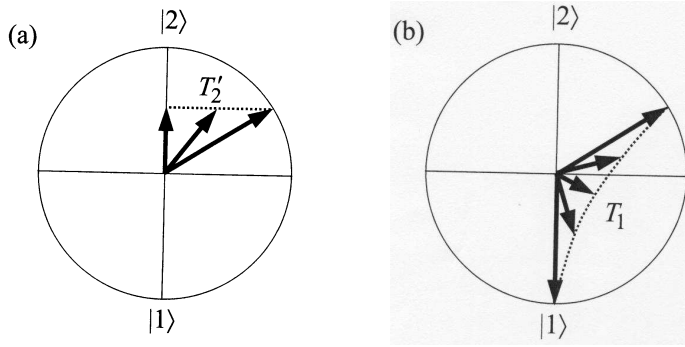


Figure 10.2: *Damping processes in the Bloch representation: (a) pure dephasing of the optical coherence at a rate $1/T_2'$, (b) longitudinal relaxation of the population at a rate $1/T_1$. The total decay rate of the optical coherence is $1/T_2 = 1/(2T_1) + 1/T_2'$. (Fig. 9.10 of ref. [FOX06])*

10.4 Strong excitation and Rabi oscillations

If the optical excitation is sufficiently strong, as compared to the atomic decay rate $\gamma_{\parallel}$ and $\gamma_{\perp}$, such that a sizeable fraction of the upper-level population is excited, various coherence and saturation effects show up. These are most conveniently described in the Bloch vector picture. We again consider the evolution of a two-level system under the influence of a monochromatic driving field, as given by Eq. (10.2), but introduce a rotating frame that differs from the one chosen in Eq. (10.1) to include the possibility of off-resonance excitation. We write the quantum state as

$$(10.13) \quad |\psi(t)\rangle = e^{-i(\omega_1 + \omega_2)t/(2\hbar)} \left[c_1(t)e^{+i\omega t/(2\hbar)}|1\rangle + c_2(t)e^{-i\omega t/(2\hbar)}|2\rangle \right] ,$$

which reduces to the former Eq. (10.1) at zero detuning ($\delta\omega \equiv \omega - \omega_0$). Schrödinger's equation now translates into the matrix description

$$(10.14) \quad \frac{d}{dt} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2}i \begin{pmatrix} -\delta\omega & 2\Omega_r \cos \omega t e^{i\omega t} \\ 2\Omega_r \cos \omega t e^{-i\omega t} & \delta\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \approx \frac{1}{2}i \begin{pmatrix} -\delta\omega & \Omega_r \\ \Omega_r & \delta\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

In the final step we neglected a fast oscillating $\cos 2\omega t$ term in the so-called **rotating-wave approximation**. The dynamics of the state amplitudes are determined by the eigenvalues of the evolution matrix. The factor $\frac{1}{2}$ disappears in the transition to the state populations $|c_i|^2$ and the coherence $c_1^* c_2$ and in the final expression for the (off-resonant) **Rabi frequency** $\Omega = \sqrt{\Omega_R^2 + \delta\omega^2}$.

The evolution of the quantum state is most easily visualized by expressing Eq. (10.14) in terms of the coefficients of the Bloch vector. In the absence of damping, one obtains the following relatively simple result

$$(10.15) \quad \frac{d}{dt} \mathbf{S} = -\boldsymbol{\omega} \times \mathbf{S},$$

where $\boldsymbol{\omega} = (\Omega_R, 0, \delta\omega)$. The resulting Rabi oscillations are simple rotations of the Bloch vector around a fixed axis, which lies perpendicular to the S_z axis for resonant excitation, but points in a different direction in case of frequency detuning. Inclusion of transverse and longitudinal damping yields the following complete description

$$(10.16) \quad \frac{d}{dt} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} -\gamma_\perp & \delta\omega & 0 \\ -\delta\omega & -\gamma_\perp & \Omega_R \\ 0 & -\Omega_R & -\gamma_\parallel \end{pmatrix} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \gamma_\parallel \bar{S}_{z,0} \end{pmatrix},$$

where $\bar{S}_{z,0} = -1$ is the equilibrium in the absence of light.

Optical excitation with a sufficiently strong and short ($t_{\text{pulse}} < T_2$) pulse provides a convenient tool to modify the quantum state of a two-level system. Eq. (10.15) indicates how resonantly-tuned optical pulses can rotate the Bloch vector over a **tipping angle** or **pulse area**

$$(10.17) \quad \Theta = \frac{\mu_{12}}{\hbar} \int_{-\infty}^{\infty} E_0(t) dt.$$

Under resonant excitation, the rotation can be around the S_x or S_y axis, depending on the phase of the optical field, i.e., the $\cos \omega t$ or $\sin \omega t$ character of the excitation. Convenient pulse areas to use are $\pi/2$, π , and 2π pulses. Starting from the ground state, π pulses produce a complete inversion of the population, while 2π pulses return the population to the ground state. The later phenomenon is called **self-induced transparency**.

Next we consider optical excitation with a continuous (optical field and analyze the effect of **optical saturation** on the steady state. After turn-on and the initial Rabi oscillations, the atomic state reaches an equilibrium that depends on the interaction strength Ω_R and detuning $\delta\omega$ in relation to the damping rates $\gamma_{\parallel}$ and $\gamma_{\perp}$. The equilibrium state found from Eq. (10.16) has

$$(10.18) \quad \bar{S}_z = \frac{\bar{S}_{z,0}}{1 + \Omega_R^2/\Omega_{\text{sat}}^2},$$

where $\Omega_{\text{sat}}^2 = (\delta\omega^2 + \gamma_{\perp}^2)\gamma_{\parallel}/\gamma_{\perp}$ quantifies the saturation effect. We singled out the z -component of the Bloch vector as this component quantifies the population inversion and codetermines the optical absorption. The absorbed intensity scales as

$$(10.19) \quad I_{\text{abs}} \propto \frac{-\bar{S}_z I}{\delta\omega^2 + \gamma_{\perp}^2} \propto \frac{I}{1 + \Delta^2 + I/I_{\text{sat}}},$$

where the normalized detuning $\Delta = \delta\omega/\gamma_{\perp}$ and $I/I_{\text{sat}} = \Omega_R^2/(\gamma_{\parallel}\gamma_{\perp})$. Optical saturation results in a reduction of the absorbed fraction I_{abs}/I and an increase in the spectral width of the absorption line; the latter phenomenon is called **power broadening**.

Rabi oscillations have been observed in many experiments, ranging from the early observation to self-induced transparency to the direct observation of damping Rabi oscillations in the atomic population and coherence. Figure 10.3 demonstrates the appearance of a **Mollow triplet** in resonantly-excited fluorescence. The appearance of these spectral side bands can be explained either in terms of a beating between the optical transition and the Rabi oscillation or in terms of dressed states, which combine the quantum description of the atom and the light field.

Appendix E of the book of Fox [FOX06] describes how the Bloch model of light-atom interaction was adapted from the Bloch model of nuclear magnetic resonance (NMR). The later model considers the evolution of a magnetic dipole vector in the presence of a static magnetic field, which leads to a Zeeman splitting of the atomic states, and a resonance rf (= radio frequency) field, which couples these levels. The comparison is most easily understood for the transition between the $(M = -\frac{1}{2}) \leftrightarrow (M = +\frac{1}{2})$ states, but also applies to other transitions. The field of quantum optics has profited a lot from techniques developed in NMR, and ESR (electron spin resonance) and, more recently, from the imaging techniques developed in MRI (magnetic resonance imaging).

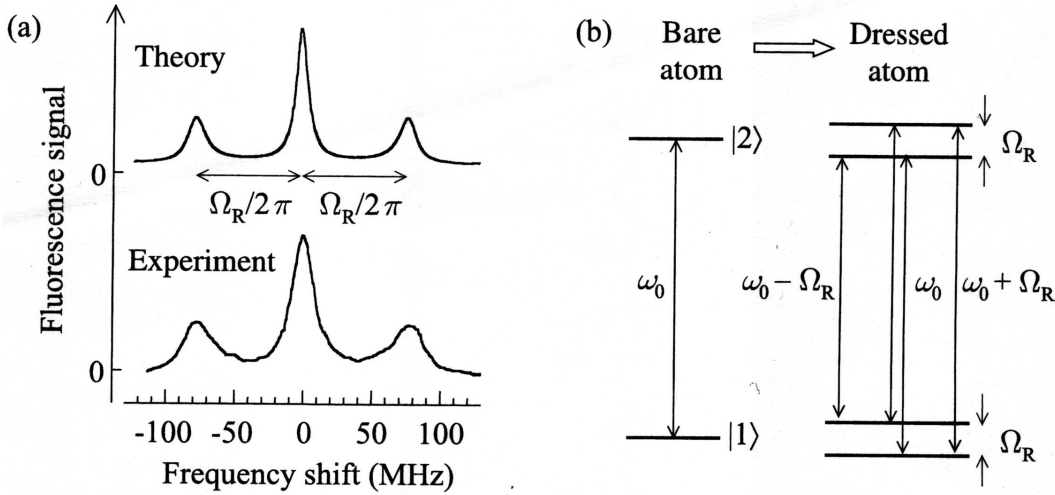


Figure 10.3: Rabi oscillations in the atomic transition can lead to a spectral splitting of the fluorescence spectrum. The resulting three-peaked structure is known as the Mollow triplet, after B.F. Mollow who predicted this phenomenon in 1969. Figure (b) shows an explanation of the Mollow triplet using the dressed atom picture. The AC Stark interaction between a two-level atom and an intense resonant light field splits the bare atom states into doublets of dressed states separated by the Rabi frequency Ω_R . (Fig. 9.7 of ref. [FOX06])

10.5 Many-level system

The two-level description that we have used so far can be a gross simplification of realistic optical transitions, which can potentially link a manifold of g_1 frequency-degenerate lower levels with g_2 frequency-degenerate upper levels. The optical transitions in these three- or more-level systems is to a large extent determined by the polarization of the optical field, the transition matrix elements $\mu_{ij} \equiv -e\langle\phi_i|\mathbf{r}|\phi_j\rangle$, and the associated selection rules. **Spin-selective excitation** is a natural consequence of these selection rules. Depending on the optical polarization, some (coherent superpositions of) levels might be optically active while others are effectively decoupled from the radiation and act as dark states. **Population trapping** due to **optical pumping** occurs when one of the dark states is a linear superposition of ground-state levels that can trap a sizeable fraction of the atomic population.

A powerful tool to manipulate multiple-level systems is the so-called **(stimulated) Raman transition**, where the atomic coherences and populations are modified by the simultaneous application of two optical fields at frequencies ω_1 and ω_2 . The joint optical interaction with a common third level results in an effective coupling between the two levels at a frequency $\omega_1 - \omega_2$. This technique has among

others been used to prepare special coherent superpositions of levels and to perform a CNOT quantum operation on atoms in a linear optical trap. It has also been used to study **electromagnetically-induced transparency** (EIT), where the presence of a strong optical field completely modifies the propagation of a second optical field at a different frequency up to the point where the speed of light is reduced to a few m/s and the light is virtually stopped! Electromagnetically-induced transparency can modify the speed of light so drastically because it induces a sharp transmission peak in the Lorentzian absorption line of the original two-level resonance (see Fig. 10.5). The link between the related imaginary and real parts of the dielectric constant is commonly known as the **Kramers-Kronig relation**.

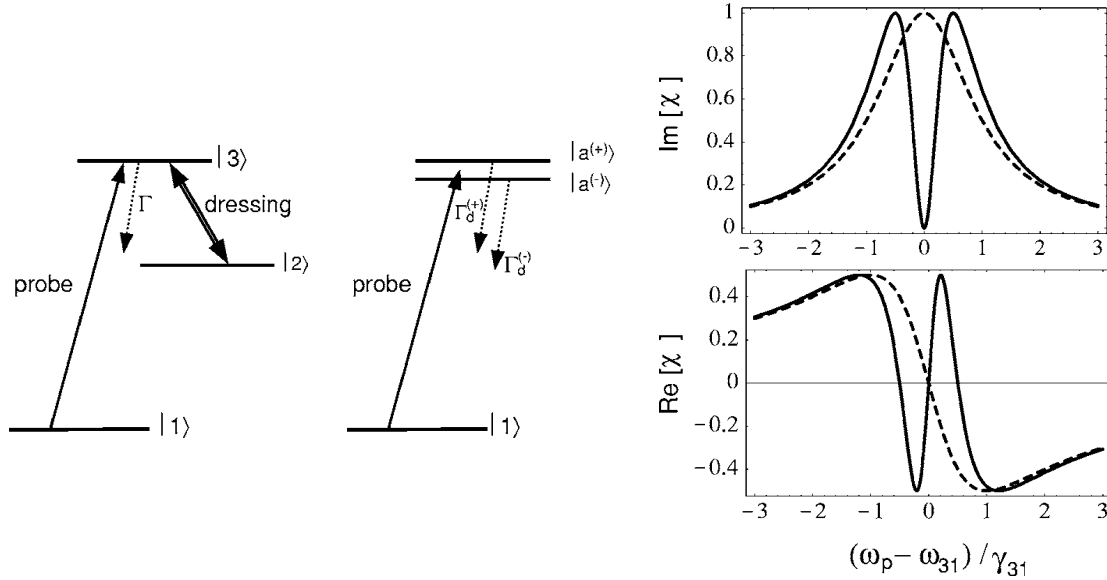


Figure 10.4: We consider the optical transmission of a weak probe laser through a medium of three-level atoms with ground state $|1\rangle$, excited state $|3\rangle$ and meta-stable state $|2\rangle$. In the absence of a second laser, the optical transmission around the probe frequency $\omega_p \approx \omega_{31}$ has a Lorentzian shape with a width γ_{31} and an associated Lorentzian dispersion profile (dashed curves in righthand figure). This transmission can be strongly modified by the presence of a second (pump/dressing) laser that drives the other $3 \leftrightarrow 2$ transition. The solid curves demonstrates the existence of **electromagnetically induced transparency** and the associated steep variation in refractive index that yields group velocities $v_g \ll c$. The middle figure explains the EIT in terms of destructive interference of the excitation pathways to the doublet of dressed states $|a^\pm\rangle = (|3\rangle \pm |2\rangle)/\sqrt{2}$ (Figs. 4 and 1 of ref. [FLE05])

Chapter 11

Atoms in optical cavities

This material is based on chapters 10 of *Quantum Optics* [FOX06]. It is supplemented with a more extensive discussion of the Jaynes-Cummings model and the Maxwell-Bloch equations.

11.1 Decay and coupling rates

We consider the dynamics of a single two-level atom located inside an optical cavity that can temporarily store part of the emitted radiation, before leaking it to free space (see Fig. 11.1). This dynamics is described by three decay rates:

- **Photon decay rate** $\kappa = 1/\tau_{\text{cav}}$ is the decay rate of the intra-cavity intensity. For a symmetric cavity of length L and constant refractive index n , comprising mirrors with intensity reflectivity $R_1 = R_2 = R$ (with $1 - R \ll 1$), the loss rate $\kappa \equiv \omega/Q = (c/nL)(1 - R)$. The quality factor $Q = \omega/\Delta\omega$ compares the FWHM spectral width of the cavity mode $\Delta\omega = \kappa$ with the optical frequency. The finesse $F \equiv \omega_{\text{FSR}}/\Delta\omega$ compares it with the so-called **free-spectral range** ω_{FSR} , being the frequency spacing between consecutive longitudinal cavity modes. In the absence of frequency dispersion, i.e. for constant refractive index n , $Q/F = 2nL/\lambda_0$.
- **(non-resonant) Atom decay rate** $\gamma = 1/T_2$ is the decay rate of the atomic coherence due to the non-resonant cavity modes only. We generally assume radiative decay to dominate over pure dephasing, making $\gamma = \gamma_{\parallel}/2$. As the solid angle $\Delta\Omega$ subtended by the resonant cavity mode is generally small ($\Delta\Omega \ll 4\pi$), the non-resonant population decay is approximately equal to its free-space value: $\gamma_{\parallel} \approx A_{21}$.

- **Atom-photon coupling rate** g_0 is the coupling rate between the atomic coherence and the intra-cavity optical field. It generally specifies the coupling rate between the dipole of a single atom and the vacuum intra-cavity field. The atom-photon coupling rate increases by a factor $\sqrt{N}$ for N identical atoms.

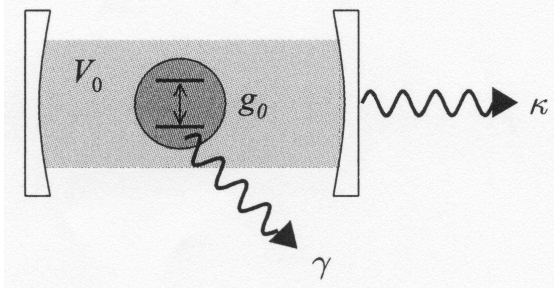


Figure 11.1: A two-level atom in a resonant cavity with modal volume V_0 . The combined system is described by three parameters: κ (photon decay rate from cavity), γ (non-resonant decay rate of atomic coherence), and g_0 (atom-cavity coupling rate). (Fig. 10.4 of ref. [FOX06])

11.2 Different coupling regimes

Quantum optics teaches us that spontaneous optical emission is not an inherent property of an emitting atom (or molecule or solid-state transition) but is co-determined by its optical environment. The spontaneous emission rate of an atom in an optical cavity can thus be either enhanced or suppressed by the modifications of the electro-magnetic mode spectrum imposed by the presence of the optical cavity. Enhanced spontaneous emission can be understood relatively easily as preferential decay into the resonant optical cavity mode. Suppressed spontaneous emission is more subtle. The required reduction of the optical density of states can be realized by placing the atom inside a (periodic) medium with a so-called **photonic bandgap**, being a frequency range over which the medium simply doesn't support optical fields. When the atom-field coupling is strong, more intriguing processes occur, such as the periodic exchange of energy between light and matter. The study of the atom-field coupling in the regime where the spontaneous emission differs substantially from its free-space behavior is called **cavity quantum electro dynamics**, or **cavity QED**.

The behavior of coupled atom-photon systems can be classified in three regimes of operation: weak, intermediate, and strong coupling. In the **weak-coupling regime** $g_0 < \{\kappa, \gamma\}$ the spontaneous emission rate of the atom in the cavity is approximately equal to its free-space value, but the angular distribution of the emission is generally modified by the presence of reflecting surfaces.

In the **intermediate-coupling regime** (sometimes also called weak-coupling), where $\kappa > g_0 > \gamma$, the coupling of the atom to the selected cavity mode can be strong

enough to dominate the radiative decay of the atom and considerably enhance this decay rate as compared to its free-space value. This so-called Purcell enhancement occurs when $g_0^2 \gg \kappa\gamma$. The regime $\gamma > g_0 > \kappa$ is rarely encountered, as this requires extremely low-loss mirrors.

The **strong-coupling regime** $g_0 > \{\kappa, \gamma\}$ is characterized by an oscillatory exchange of energy between the atomic and photon system at the so-called **vacuum Rabi frequency** $\Omega^{\text{vac}} = 2g_0$. In this regime, the coupling between the emitting atom and the optical field is so strong that the optical radiation emitted by the atom into the optical cavity mode can be reabsorbed and re-emitted several times before it finally escapes through the cavity mirrors. This periodic emission and absorption is visible in the optical spectrum as a splitting of the resonance into a doublet of resonances with mixed atom-field properties spaced by the mentioned vacuum Rabi splitting.

11.3 Intermediate coupling: Purcell effect

In the intermediate coupling regime, where $\kappa > g_0 > \gamma$, the spontaneous emission in the selected cavity mode can be as large or even larger than the spontaneous emission in all other (non-resonant) optical modes. The **Purcell factor** F_P compares the decay rate of an atom in a resonant cavity with that of the same atom in free space. It can be defined in two different ways, where either $F_P = 0$ or $F_P = 1$ without a cavity. Fox chooses to define

$$(11.1) \quad F_P \equiv \frac{\text{Decay rate of atom in cavity mode}}{\text{Decay rate of atom in free space}} = (3\times)(2\times)Q \frac{(\lambda_0/n)^3}{4\pi^2 V_0},$$

where $Q \equiv \omega/\Delta\omega$ is the quality factor of the cavity mode and V_0 is the modal volume. The factor $(3\times)$ should only be included when the atomic dipole is aligned with the polarization of the intra-cavity field; it results from the free-space average $\langle \cos^2(\theta) \rangle = 1/3$. The factor $(2\times)$ should only be included when the atomic dipole is positioned in an anti-node of the (standing-wave) intra-cavity field and when the mode volume V_0 is defined on the basis of the positioned-averaged field, being averaged over possible nodes and anti-nodes.

The derivation of Eq. (11.1) is based on the application of Fermi's golden rule

$$(11.2) \quad W = \frac{2\pi}{\hbar^2} |\boldsymbol{\mu} \cdot \mathbf{E}|^2 g(\omega),$$

to the two mentioned emission rates. For free-space emission we combine the general result $|\mathbf{E}_{\text{vac}}|^2 = \hbar\omega/(2\epsilon_0 V)$ per mode with a mode density $g(\omega) = V\omega^2/(\pi^2 c^3)$ per unit radial bandwidth, where V is an arbitrary quantization volume that drops

out of the description. For emission into the cavity mode we combine a similar result $|\mathbf{E}|^2 = \hbar\omega/(2\epsilon_0 V_0)$, where V_0 is now a fixed **modal volume**, with the mode density $g(\omega) = 2\Delta\omega_c/\pi$ obtained by spreading the intensity of this single mode over a Lorentzian spectrum with a FWHM of $\Delta\omega_c$. The result of this calculation is the mentioned

$$(11.3) \quad F_P = \frac{[\hbar\omega/(2\epsilon_0 V_0)]}{[\hbar\omega/(2\epsilon_0)]} \frac{2\Delta\omega_c/\pi}{\omega^2/(\pi^2\omega^3)} = Q \frac{(\lambda_0/n)^3}{4\pi^2 V_0},$$

for a randomly oriented dipole. Extra factors of $(3\times)$ and $(2\times)$ appear for an oriented dipole positioned in an anti-node of the field. Figure 11.2 shows the original article from 1946 in which Purcell discusses this effect [PUR46]. This short paper, which is actually part of a conference proceedings, has been cited approximately 1600 times!

B10. Spontaneous Emission Probabilities at Radio Frequencies. E. M. PURCELL, *Harvard University*.—For nuclear magnetic moment transitions at radio frequencies the probability of spontaneous emission, computed from

$$A_\nu = (8\pi\nu^2/c^3)\hbar\nu(8\pi^3\mu^2/3h^2) \text{ sec.}^{-1},$$

Figure 11.2: Copy of the original paper of Purcell from 1946, in which he discusses spontaneous emission decay at radio frequencies and notes that the spontaneous decay of an aligned nuclear spin, which is incredibly slow in free space, can be enhanced by many orders of magnitude in a resonant cavity [PUR46].

is so small that this process is not effective in bringing a spin system into thermal equilibrium with its surroundings. At 300°K, for $\nu=10^7 \text{ sec.}^{-1}$, $\mu=1$ nuclear magneton, the corresponding relaxation time would be 5×10^{21} seconds! However, for a system coupled to a resonant electrical circuit, the factor $8\pi\nu^2/c^3$ no longer gives correctly the number of radiation oscillators per unit volume, in unit frequency range, there being now *one* oscillator in the frequency range ν/Q associated with the circuit. The spontaneous emission probability is thereby increased, and the relaxation time reduced, by a factor $f=3Q\lambda^3/4\pi^2 V$, where V is the volume of the resonator. If a is a dimension characteristic of the circuit so that $V\sim a^3$, and if δ is the skin-depth at frequency ν , $f\sim\lambda^3/a^2\delta$. For a non-resonant circuit $f\sim\lambda^3/a^3$, and for $a<\delta$ it can be shown that $f\sim\lambda^3/a\delta^2$. If small metallic particles, of diameter 10^{-3} cm are mixed with a nuclear-magnetic medium at room temperature, spontaneous emission should establish thermal equilibrium in a time of the order of minutes, for $\nu=10^7 \text{ sec.}^{-1}$.

I also like to present an alternative and more direct derivation of the Purcell factor, which is based on the notion that spontaneous emission is induced by vacuum fluctuations that leak in from all directions [EXT96]. Consider a cavity composed of two highly-reflecting mirrors that together extend a solid angle $\Delta\Omega \ll 4\pi$. Vacuum fluctuations that leak in via the mirrors will bounce up and down between these mirrors to build up a field of strongly increased intra-cavity intensity. The intensity in the anti-nodes of this standing wave is enhanced by a factor $4/(1-R)$ with respect

to the incident intensity, where R is the intensity reflectivity. The spontaneous emission rate of an atom positioned in an anti-nodes emitting in the direction of one of the mirrors will be enhanced by the same factor. This line of reasoning yields a Purcell factor

$$(11.4) \quad F_P = (3\times)(2\times) \frac{\Delta\Omega}{4\pi} \frac{2}{1-R},$$

where the factors $(3\times)$ and $(2\times)$ again refer to dipole alignment and positioning, respectively.

Although Eqs. (11.3) and (11.4) look entirely different, they are actually the same on account of two relations. First of all, diffraction links the opening angle $\Delta\Omega$ to the minimum beam area A by the Fourier relation $A\Delta\Omega \approx (\lambda_0/n)^2$ (related to the optical entendue). Second, the quality factor of the cavity mode is $Q = \omega/\Delta\omega_{\text{cav}} = 2\pi Ln/[\lambda_0(1-R)]$ in the absence of dispersion. These two relations make Eqs. (11.3) and (11.4) identical, if we define the modal volume as $V_0 = AL$.

11.4 Strong coupling: vacuum Rabi splitting

The strong coupling regime $g_0 > \{\kappa, \gamma\}$ is characterized by a continuous exchange of the coherent excitation between the atom and optical field. This phenomenon is best understood by starting from the so-called Jaynes-Cumming Hamiltonian

$$(11.5) \quad \hat{H} = \hat{H}_{\text{field}} + \hat{H}_{\text{atom}} + \hat{H}_{\text{int}} = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) + \hbar\omega_0\hat{S}_z + i\hbar g_0(\hat{S}_-\hat{a}^\dagger - \hat{S}_+\hat{a}),$$

which describes the combined atom-field evolution in the absence of damping. In this Hamiltonian, $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the intra-cavity optical field. The atomic population is described with the operator $\hat{S}_z = \frac{1}{2}(|e\rangle\langle e| - |g\rangle\langle g|)$, and the atomic coherence with the raising operator $\hat{S}_+ = |e\rangle\langle g|$ and its Hermitian conjugate $\hat{S}_- = |g\rangle\langle e|$, where $|e\rangle$ and $|g\rangle$ are the excited and ground state, respectively. The atomic part of the Hamiltonian $\hbar\omega_0\hat{S}_z$ is simply proportional to the population inversion.

The atom-field interaction Hamiltonian $\hat{H}_{\text{int}}$ is the quantum-mechanical equivalent of the electric-dipole interaction energy

$$(11.6) \quad \hat{H}_{\text{int}} = -\hat{\mu}\hat{E} \rightarrow i\hbar g_0(\hat{S}_-\hat{a}^\dagger - \hat{S}_+\hat{a}).$$

The derivation from the second to the third expression involves the following steps. The dipole operator in Eq. (11.6) is written as

$$(11.7) \quad \hat{n}u = \mu_{eg}\hat{S}_+ + \mu_{ge}\hat{S}_- = \mu_{21}\hat{S}_+ + \mu_{12}\hat{S}_-.$$

where $\mu_{12} \equiv e\langle 1|x|2\rangle$ is the transition dipole moment. The intra-cavity field is written in its operator form

$$(11.8) \quad \hat{\mathbf{E}}(\mathbf{r}, t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0}} (\hat{a}e^{-i\omega_c t}\mathbf{F}(\mathbf{r}) - \hat{a}^\dagger e^{i\omega_c t}\mathbf{F}^*(\mathbf{r})) ,$$

where the mode profile $\mathbf{F}(\mathbf{r})$ is normalized via

$$(11.9) \quad \int |\mathbf{F}(\mathbf{r})|^2 dx dy dz = 1 ,$$

The spatially-averaged mode profile $\langle |\mathbf{F}(\mathbf{r})|^2 \rangle_{\mathbf{r}} \approx 1/V_0$ when averaged over the nodes and anti-nodes of the field. Combination of these factors gave the earlier result Eq. (7.4)

$$(11.10) \quad \langle E_{\text{vac}}^2 \rangle = \frac{\hbar\omega}{2\epsilon_0 V_0} ,$$

The factor i in Eq. (11.8) originates from the classical relation between the electric field and the vector field $\mathbf{E} = -d\mathbf{A}/dt$. A combination of these equations yields the atom-field coupling rate

$$(11.11) \quad g_0 = \xi\mu_{12}E_{\text{vac}}/\hbar = \xi\mu_{12}\sqrt{\frac{\omega}{2\hbar\epsilon_0 V}} ,$$

where the factor $\xi = \cos\theta$ accounts for the dipole orientation factor with respect to the optical polarization of the cavity mode ($\xi = 1$ for aligned dipoles). Note that in the final step of Eq. (11.6), the Jaynes-Cummings model only keeps the two co-rotating terms and discards the two counter-rotating terms. These counter-rotating terms average out as they oscillate at a frequency 2ω . They correspond to the strange process $\hat{S}_-\hat{a} = |g\rangle\langle e|\hat{a}$, which annihilates a photon upon population decay, and $\hat{S}_+\hat{a}^\dagger$, which creates a photon while raising the atom population from $|g\rangle$ to $|e\rangle$.

The eigenstates of the Jaynes-Cummings Hamiltonian are the so-called **dressed states**

$$(11.12) \quad |\Psi_n^\pm\rangle = \frac{1}{\sqrt{2}} (|g; n\rangle \pm |e; n-1\rangle) ,$$

with energy $E_n^\pm = (n+\frac{1}{2})\hbar\omega \pm \sqrt{n}\hbar g_0$ (see Fig. 11.3). The frequency splitting between the pair of dressed states with the lowest energy is the so-called **vacuum Rabi splitting** $\Omega^{\text{vac}} = 2g_0$. The frequency splitting between pairs of dressed states higher up on the Jaynes-Cummings ladder increases as $\sqrt{n}$ for the transition $n \leftrightarrow (n-1)$ photons. Another factor $\sqrt{N}$ appears when the model is extended to describe a cavity with N active atoms.

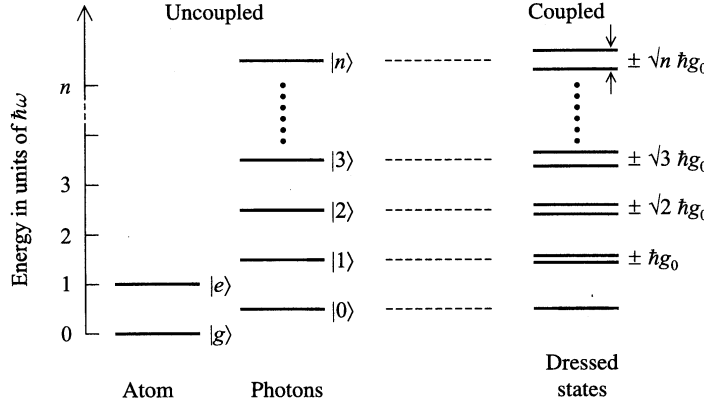


Figure 11.3: The Jaynes-Cummings ladder describes the states of a coupled atom-photon system with a coupling constant g_0 . (Fig. 10.9 of ref. [FOX06])

11.5 Maxwell-Bloch equations

Unfortunately, different authors use different definitions for the three important rates κ, γ, g_0 . The loss rate of the intra-cavity field is for instance generally defined as an amplitude decay rate instead of an intensity decay rate. We will do so in this final section, using the symbol $\tilde{\kappa} = \kappa/2$ to indicate the amplitude decay rate. Some authors define the atom decay rate as the decay of the atomic population instead of its polarization, using $\gamma_{\parallel}$ instead of $\gamma_{\perp}$. Other complications arise if the atomic transition frequency fluctuates on account of a time-varying environment. This effect can be characterized by a pure dephasing rate γ^* , where $\gamma_{\perp} = \gamma^* + \gamma_{\parallel}/2$. We will not consider this possibility any further.

The coupled atom-field dynamics is driven by the Hamiltonian of Eq. (11.5) supplemented by two non-Hermitian terms that describe the loss of the cavity field and the combined loss of atomic population and inversion. We consider only the resonant case, where the cavity is tuned to the transition frequency of atomic system ($\omega_c = \omega_a$), and use slowly-varying amplitudes for the operators $\hat{a}$ and $\hat{S}_{-}$. The combined atom-field dynamics can then be rephrased in the so-called **Maxwell-Bloch equations** [AUF07]

$$(11.13) \quad \frac{d}{dt}\hat{a} = -\tilde{\kappa}\hat{a} - g_0\hat{S}_{-} + i\sqrt{2\tilde{\kappa}}\hat{b}_{\text{in}},$$

$$(11.14) \quad \frac{d}{dt}\hat{S}_{-} = -2\hat{S}_z g_0\hat{a} - \gamma_{\perp}\hat{S}_{-},$$

$$(11.15) \quad \frac{d}{dt}\hat{S}_z = g_0\left(\hat{S}_{+}\hat{a} + \hat{a}^{\dagger}\hat{S}_{-}\right) - \gamma_{\parallel}(\hat{S}_z + \tfrac{1}{2}).$$

The operator $\hat{b}_{\text{in}}$ represents an external optical field that can be coupling into the cavity and will be reflected as $\hat{b}_r = \hat{b}_{\text{in}} + i\sqrt{2\tilde{\kappa}}\hat{a}$. At sufficiently weak excitation, the excited-state fraction remains negligible at $-2\langle\hat{S}_z\rangle \approx 1$, and the first two equations

simplify to

$$(11.16) \quad \frac{d}{dt}\hat{a} = -\tilde{\kappa}\hat{a} - g_0\hat{S}_-,$$

$$(11.17) \quad \frac{d}{dt}\hat{S}_- = g_0\hat{a} - \gamma_\perp\hat{S}_-,$$

in the absence of input. The dynamics of this coupled system is characterized by two exponential forms $\exp(\lambda_\pm t)$ with eigenvalues

$$(11.18) \quad \lambda_\pm = -\frac{1}{2}(\tilde{\kappa} + \gamma_\perp) \pm \sqrt{\frac{1}{4}(\tilde{\kappa} - \gamma_\perp)^2 - g_0^2}.$$

The three different coupling regimes that we mentioned at the start of this chapter can be easily recognized in these eigenvalues. For a weakly coupled system with $g_0 \ll \{\tilde{\kappa}, \gamma_\perp\}$ we obtain $\lambda_- \approx -\tilde{\kappa}$ and $\lambda_+ \approx -\gamma_\perp$ as the original decay rates of the optical field amplitude and the atomic coherence. In the intermediate-coupling regime, where $\tilde{\kappa} \ll g_0 \ll \gamma_\perp$, the eigenvalues are equal to $\lambda_- \approx -\tilde{\kappa}$ and $\lambda_+ \approx -(\gamma_\perp + g_0^2/\tilde{\kappa})$. The spectrum now contains two peaks with a very different character: a wide peak associated with the optical cavity resonance and a more narrow peak associated with the atomic resonance. The width of the latter is enhanced by the Purcell effect from $\gamma_\perp$ to $\gamma_\perp + g_0^2/\tilde{\kappa}$. The associated Purcell factor is

$$(11.19) \quad F_P = g_0^2/(\tilde{\kappa}\gamma_\perp) = 2g_0^2/(\kappa\gamma).$$

For completeness, we note that the amount of coupling in the intermediate regime can equivalently be quantified by the critical atom number $N_0 \equiv 2\kappa\gamma/g_0^2$ or its inverse the cooperativity parameter $C \equiv 1/N_0$. In the strong-coupling regime ($g_0 \gg \tilde{\kappa}, \gamma_\perp$), the eigenvalues $\lambda_\pm \approx -\frac{1}{2}(\tilde{\kappa} + \gamma_\perp) \pm ig_0$ are complex and hence induce frequency shifts and oscillatory behavior. The optical spectrum now contains two equally strong peaks at a mutual distance $\Omega_{\text{vac}} = 2g_0$. The damping rate of these combined atom-field excitations are simply given by the average of the field and atom damping.

The vacuum Rabi oscillations has a distinct quantum-mechanical flavor that seems to defy a classical description. However, a relatively simple classical explanation has been found. This explanation is based on the idea that the intra-cavity atom does not only absorb light, but also modifies its propagation. The associated frequency-dependent refractive index $n_r(\omega)$ will shift the optical resonance frequency of the cavity and can even split it into two separate transmission peaks if the coupling is strong enough. A derivation of this effect combines the resonance condition of the cavity modes $n_r(\omega)\omega = m\pi c/L$, which restricts the round-trip optical path length to an integer number of m optical wavelengths, with the Lorentzian expression for the complex refractive index of the medium.[ZHU90] In the strong-coupling regime, the resonant optical absorption of even a single atom in the cavity is stronger than

the optical losses through the cavity mirrors, such that it suppresses the central resonance. Only the two dispersion-shifted outer resonances are now visible. As these resonances are detuned by $\pm g_0$, they experience less atomic absorption (see Fig. 11.4).

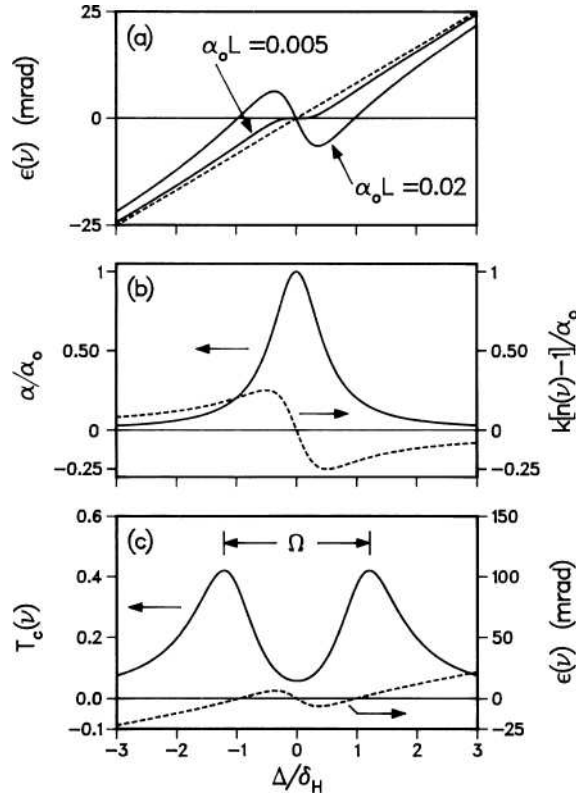


Figure 11.4: (a) Phase shift experienced by the field upon completion of a round trip through the cavity for various values of the line center single pass absorption as a function of the normalized detuning (Δ/δ_H in figures). (b) Normalized absorption (solid line) and change in refractive index (dashed line) produced by Lorentz oscillator. (c) Cavity transmission (solid line) and phase shift (dashed line). [ZHU90]

Chapter 12

Quantum information

This material is based on chapters 12 and 14 of the book *Quantum Optics* [FOX06], supplemented with extra material from the scientific literature. I have skipped the topic of “Quantum Computation”. This topic, which is introduced in chapter 13 of the book Fox, as this topic is extensive enough to fill a course of its own (see 676-pages thick text book *Quantum Computation and Quantum Information* by Nielsen and Chuang.[NIE00]). The final chapter of this syllabus is strongly geared towards quantum optics. It introduces and discusses several intriguing key experiments in “Quantum Optics and Quantum Information”. Most of these experiments are only possible with a special form of light, comprising quantum-entangled pairs of photons.

12.1 Quantum cryptography: BB84 protocol

Chapter 12 of the book of Fox is dedicated to quantum communication. It includes an extensive discussion of the common **BB84 communication protocol**, invented in 1984 by Bennett and Brassard.[BEN84] It also discusses various technical aspects of quantum communication, most important its security against eavesdropping and its resilience against imperfections such as optical loss and birefringence of the communication channel and multi-photon emission of the source. I have nothing to add to their extensive discussion and will highlight the essential ingredients of quantum communication in a short powerpoint presentation. The only aspect that I like to stress is that the **quantum-no-cloning theorem** has a simple physical origin in optics. Amplification by stimulated emission of radiation is always accompanied by spontaneous emission. As the latter process is random it necessarily adds noise to the communication channel.

12.2 Quantum entanglement

A **quantum bit** or **qubit** is the quantum-mechanical equivalent of a classical bit in ordinary computing. Whereas a normal bit is either 0 or 1, a quantum bit can be in any linear superposition of two orthogonal 0 and 1 quantum states

$$(12.1) \quad |\psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

with complex amplitudes c_0 and c_1 . Normalization requires $|c_0|^2 + |c_1|^2 = 1$, such that each quantum bit corresponds to a point on the Bloch sphere. Each quantum operation on a single quantum bit, also denoted as a **single-qubit gate**, corresponds to a specific rotation in Hilbert space. For a **two-qubit gate** the operation on the **target** qubit 1 depends on the state of the **control** qubit 2. The most important two-qubit gate is the **(quantum) CNOT**, which inverts qubit 1 (= **NOT**) under the condition (= **C**) that qubit 2 is in a certain state.

Next we consider a composite quantum system comprising several subsystems. The state of this composite system is **quantum entangled** if it cannot be written as a direct product of quantum states of the subsystems, i.e., if

$$(12.2) \quad |\psi\rangle_{\text{tot}} \neq |\psi\rangle_1 \otimes |\psi\rangle_2 \dots$$

A composite system of two qubits is quantum entangled if the total quantum state $|\psi\rangle = c_{00}|0,0\rangle + c_{01}|0,1\rangle + c_{10}|1,0\rangle + c_{11}|1,1\rangle$ does not factorize in two states of the form of Eq. (12.1). While the Hilbert space of 2 qubits has a modest dimension of 4, the Hilbert space of a composite system of N qubits has a dimension of 2^N . Every extra quantum bit increases this dimension by a factor 2, as the additional qubit doesn't only contain information on its own quantum state, but also on its entanglement with all possible combinations of the other qubits. As a result, a quantum system of 30 qubits can in principle store as much as 1 Gigabit of classical information, whereas the potential storage capacity of a system of 100 qubits is beyond the storage capacity of all computers presently available on earth.

12.3 Quantum-entangled photon pairs

In optics, quantum entanglement can be produced relatively easily through **spontaneous parametric down-conversion (SPDC)**. [KWI95] In this nonlinear optical process, a single photon at frequency ω_p splits into a pair of photons at frequencies ω_1 and ω_2 . Energy conservation requires that these frequencies add up as $\omega_1 + \omega_2 = \omega_p$, but it doesn't restrict the individual frequencies. The combination of a conservation law for the photon pair and freedom of the individual photons

forms the origin of quantum entanglement. The quantum entanglement associated with energy conservation exists in time/frequency. Momentum conservation results in a similar quantum entanglement in position/momentum. We will not consider these two forms of entanglement, but instead concentrate on frequency-degenerate emission ($\omega_1 \approx \omega_2$) in two well-defined directions, referred to as beam 1 and beam 2. Instead, we will only consider **polarization entanglement** generated in so-called type II SPDC, where the polarization of each individual photon is random but the polarization of the pair is fixed by the generation process. More specifically, this form of SPDC generates a quantum-entangled (photon-pair) state of the form

$$(12.3) \quad |\psi\rangle = (|H_1, V_2\rangle + e^{i\varphi}|V_1, H_2\rangle) / \sqrt{2},$$

where $\{H, V\}_{\{1,2\}}$ refers to the polarization state of the photon in beam $\{1, 2\}$, respectively. The phase φ is determined by the geometry of the generation process.

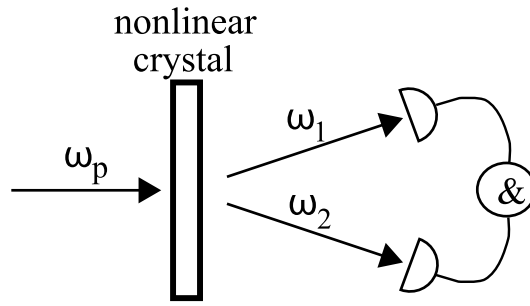


Figure 12.1: *Spontaneous parametric down-conversion (SPDC) is a nonlinear optical process where an occasional input photon at frequency ω_p spontaneously splits into a pair of down-converted photons at frequencies ω_1 and ω_2 with $\omega_1 + \omega_2 = \omega_p$. Selective detection of these photon pairs can be performed via coincidence detection, where pulses from two single photon counters are fed into a fast AND/coincidence gate, indicated by the symbol &.*

Despite the extremely low conversion efficiency from single pump photons to pairs of quantum-entangled photons, SPDC has become the workhorse in hundreds of experiments on quantum entanglement. The reason for this is two-fold. First of all, the generation process is relatively straightforward. Apart from the limited yield, the biggest experimental challenge is to keep the polarization-entangled state sufficiently pure by avoiding spatial and spectral labelling of the photons.[KWI95] Second, quantum-entangled photon pairs can be detected in a very selective way via so-called **coincidence detection** (see Fig. 12.1). Coincidence detection selects photon pairs by feeding the pulses from two single-photon counters into a fast AND gate and looking only at the coincidence events, where two pulses arrive at exactly “the same time” or at least “the same time within the experimentally-limited gate

time of typically 1 ns”. This post-selection on photon-pair detection is a very powerful tool as it removes most of the back ground signal originating from single-photon events.

12.4 Hong-Ou-Mandel interferometer

In 1987 Hong, Ou, and Mandel performed one of the key experiment with entangled photons.[HON87] They demonstrated an unusual form of two-photon interference in an experiment that combined the two beams produced by SPDC on a beam splitter and recorded the coincidence rate between detection events of two single photon counters as a function of the time delay between the two beams. Their experimental setup and key result is depicted as Fig. 12.2. At zero time delay, the two-photon interference in the HOM interferometer is such that the two photons always pair up or bunch behind the beam splitter. They either both travel to photon counter D1 or to D2, but never split up. This peculiar behavior is demonstrated by a strong reduction in the coincidence rate at zero delay. Two-photon interference does not produce interference fringes. Actually, it doesn’t even require quantum entanglement. It only requires the two input ports of the beam splitter to be populated by indistinguishable single-photon states (see Chapter 8 of this syllabus).

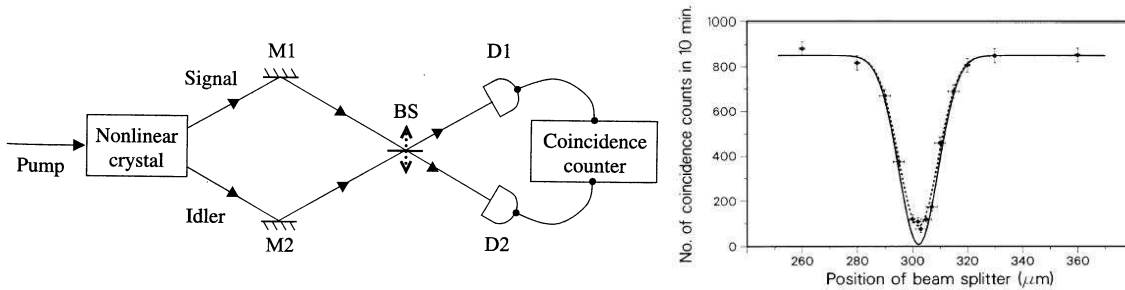


Figure 12.2: Two-photon interference in a Hong-Ou-Mandel (HOM) interferometer. The experimental setup (left) shows how two beams, generated via SPDC in a nonlinear crystal, are combined at a beam splitter. The coincidence rate between detection events of two single photon counters is recorded as a function of the time delay between both interferometer arms. The prominent dip in the righthand figure demonstrates the occurrence of photon bunching around zero delay. (figures from ref. [HON87])

Figure 12.3 demonstrates the occurrence of two-photon interference between two photons that do not originate from a single quantum-entangled pair.[RAR97] It thus demonstrates that this form of photon bunching is not due to some special

relation between the two quantum-entangled photons, but is rather due to the single-photon nature of the light.

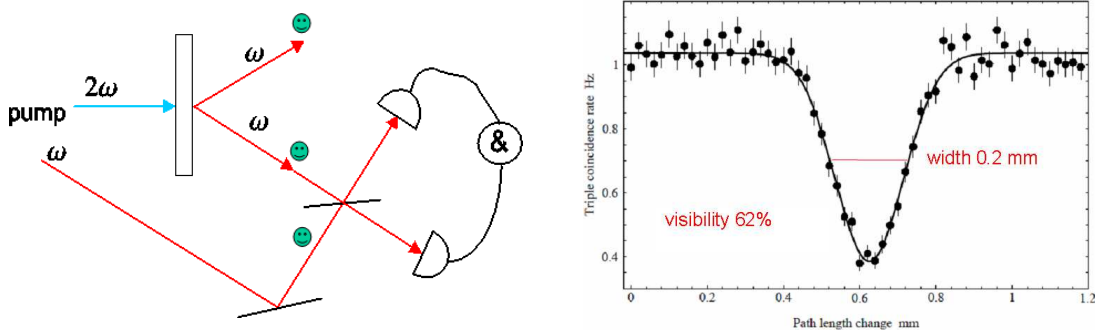


Figure 12.3: Two-photon interference between one of the SPDC beams and a weak coherent beam at half the pump frequency. The existence of a HOM dip, albeit with limited visibility, demonstrates two-photon bunching between “independent photons”. Hence, two-photon bunching does not require quantum-entanglement; single-photon emission suffices. (figures from ref. [RAR97])

Figure 12.4 shows an intriguing extension of the HOM interferometer, where the two output ports of the first beam splitter are recombined at a second beam splitter.[RAR90] The coincidence count rate recorded behind this second beam splitter oscillates rapidly as a function of the time delay in the second interferometer.[RAR90] Being caused by the interfering probability amplitudes of having a full photon pair in either the upper or the lower arm of this interferometer, the observed oscillation is in fact twice as fast as expected for individual photons. This experiment thus demonstrates the importance of the two-photon DeBroglie wavelength, being equal to the pump wavelength λ_p instead of the SPDC wavelengths $\lambda_{\{1,2\}}$. This more compact wavelength has among others been used to perform quantum imaging with quantum-entangled light below the resolution limit set for imaging with classical light.[BOT00]

12.5 Bell's inequality

In 1935 Einstein, Podolsky, and Rosen introduced a “Gedanken experiment” to demonstrate what they called the incompleteness of quantum mechanics.[EIN35] The **EPR experiment** points out that mysterious correlations exist between measurement outcomes obtained on a two-component quantum-entangled system. These correlations require a quantum-mechanical description of the full system. As they cannot be explained from the quantum-mechanical descriptions of each individual component, this description is “incomplete”. EPR speculate that the mentioned

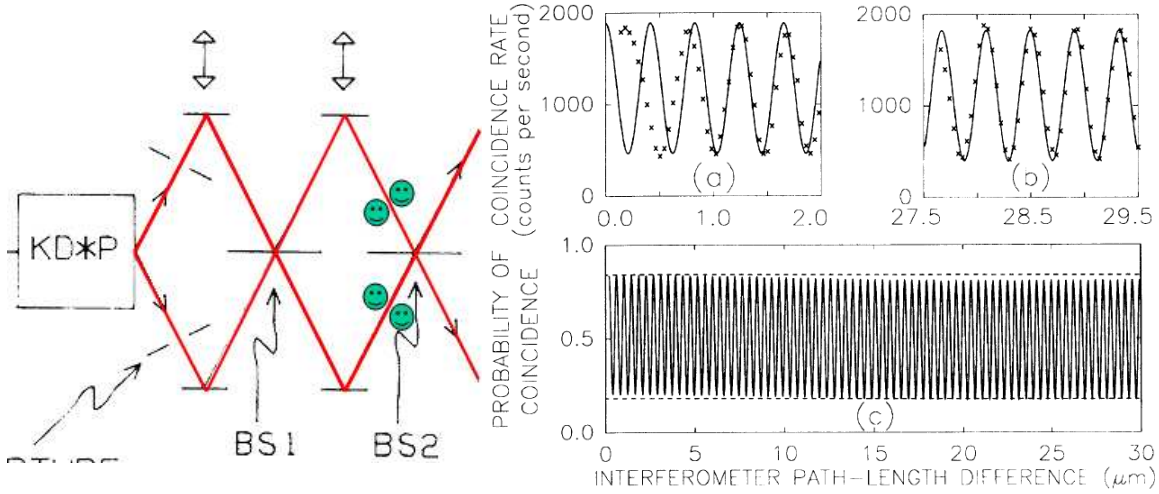


Figure 12.4: Two-photon interference behind a Hong-Ou-Mandel (HOM) interferometer. The two output ports of a HOM interferometer are combined at a second beam splitter. The coincidence rate behind this second beam splitter exhibits fast oscillations as a function of the time delay in the second interferometer, thus demonstrating the importance of the two-photon DeBroglie wavelength at the pump wavelength λ_p . The wavelength of the pump light is $\lambda_p = 413 \text{ nm}$. (figures from ref. [RAR97])

correlations could possibly be explained with a classical **hidden variable theory** that contains hidden information on the probabilistic outcomes of the measurements. It took till 1964, before Bell pointed out that no classical theory that is based on local hidden variables is able to explain the outcomes of certain series of quantum measurements.[BEL64] When we apply Bell's arguments to polarization-entangled photon pairs, Bell's inequality describes a relation between values of the function $P(\theta_1, \theta_2)$ that describes the probability to observe a photon pair behind polarizers set at polarization angles θ_1 and θ_2 in beam 1 and 2, respectively. Section 14.4.2 of the book of Fox describes the experimental implementation of Bell's inequality as proposed by Clauser, Horne, Shimony, and Holt.[CLA69] The very first experimental demonstration of Bell's inequalities was performed by Aspect et al., who produced photon pairs not via SPDC but via an atomic decay cascade in Calcium.[ASP81]

12.6 Quantum teleportation

Quantum-entangled photon pairs have played a key role in the demonstration of quantum teleportation.[BOU97] The key idea of quantum teleportation is to transfer an unknown quantum state of a particle (in our case a photon) from A to B without

transferring the particle itself. Quantum teleportation is challenging because of the quantum no-cloning theorem. This theorem states that it is impossible to clone an unknown quantum state $|\psi\rangle$, or alternatively to fully characterize an unknown quantum state and then copy it. The reason is that any ordinary measurement on $|\psi\rangle$ will provide information that inherently modifies the state. Such a modification does not occur in the quantum teleportation scheme depicted in Fig. 12.5. In quantum teleportation, the unknown state $|\psi\rangle$ is first combined with one of the particles of a two-particle quantum-entangled state before a Bell-state measurement is performed on this new two-particle system. The outcome of this pair measurement in A doesn't provide specific information on $|\psi\rangle$, but does transfer part of its quantum information to the second particle of the entangled pair in B. By communicating the outcome of the classical measurement in A to B and modifying the quantum state accordingly, the surviving particle in B acquires the (still unknown) quantum state $|\psi\rangle$. This state has thus effectively been teleported from A to B (see Fig. 12.5)

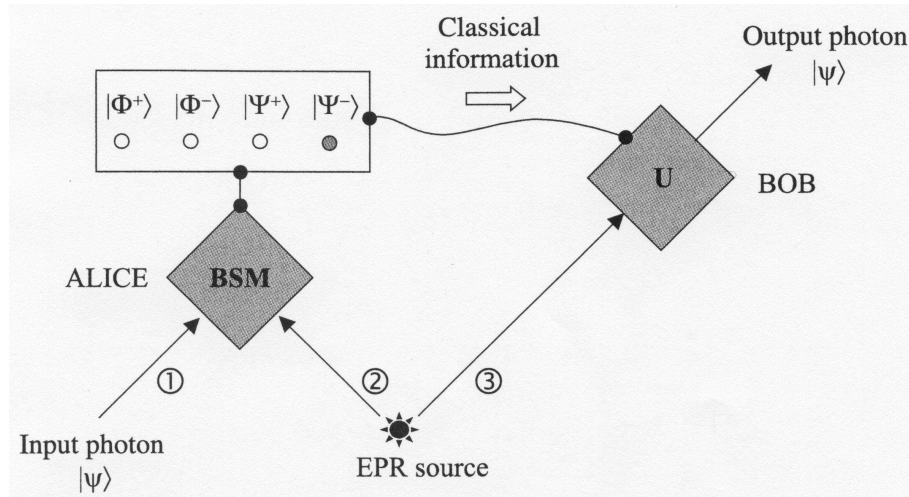


Figure 12.5: Schematic diagram for teleportation of the quantum state of a photon. The quantum state $|\psi\rangle$ of photon 1 (lower-left corner) at A(lice) can be teleported to the output photon (top-right corner) at B(ob), by (i) mixing photon 1 with photon 2 from an entangled 2-3 pair, (ii) performing a Bell state measurement on photons 1 and 2, using single photon counters and coincidence logic behind a beam splitter, (iii) communicating the classical outcome to B, and (iv) modifying photon 3 accordingly.

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